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# Nonlinear spatial evolution of helical disturbances to an axial jet

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We investigate the weakly nonlinear spatial evolution of helical disturbances of an axisymmetrical jet which are the analogue of three-dimensional disturbances, such as a single oblique wave (the wave vector is directed at an angle to the main flow velocity) in plane-parallel flows. It is shown that when a supercriticality is large enough, the perturbation amplitude $A$ grows in the streamwise direction (along $z$) explosively: $A \sim (z_0-z)^{-5/2}$, though more slowly than in the case of essentially three-dimensional disturbances in the form of a pair of oblique waves ($A \sim (z_0-z)^{-3}$; Goldstein & Choi 1989). The nonlinearity needed for such a growth, is due equally to the cylindricity of shear layer and to the spatial character of the evolution (in the temporal problem the 'evolution' contribution is absent). At a smaller supercriticality, the evolution equation has a non-local (integral in $z$) nonlinearity, unusual for the regime of a viscous critical layer. Scenarios of disturbance development for different levels of supercriticality are studied, with proper account taken of viscous broadening of the flow.

## 1. Introduction

A considerable number of papers have recently appeared devoted to the weakly nonlinear dynamics of disturbances in high-Reynolds-number shear flows in the presence of a critical layer. At first the main efforts were directed toward investigation of two-dimensional disturbances (i.e. depending only on two spatial coordinates – along directions of unperturbed velocity ($z_1$) and its gradient ($z_2$)) because, on the one hand, it is these disturbances that are most dangerous (from the viewpoint of stability loss) in an incompressible fluid and, on the other, two-dimensional problems are much simpler to solve. This ('two-dimensional') stage of theory development provided us with some fruitful ideas and methods, as well as with some important results that are worth at least outlining.

'Weakly nonlinear theory at high Reynolds numbers' means that weakly supercritical low-amplitude perturbations in almost inviscid flows are studied. More strictly, this means that perturbation amplitude $A$, its growth rate $\gamma = |A^{-1}\,\mathrm{d}A/\mathrm{d}\xi|$ and the inverse Reynolds number $\nu$ of the unperturbed flow are small compared with unity (here $\xi$ is the evolution variable, i.e. time $t$ or streamwise coordinate $z_1$, depending on the problem under consideration; all quantities are scaled by a characteristic velocity and the half-width of flow). In view of this the spatial structure of a perturbation in the main part of flow (out of the so-called critical layer) is reasonably well described by the neutral mode of inviscid linear theory, whereas viscosity, supercriticality and nonlinearity provide only small corrections to this structure, but it is these factors that determine the form of the evolution equation

$$\frac{dA}{d\xi} = F(A), \quad F(0) = 0 \tag{1.1}$$

which describes the amplitude development in $\xi$.

The critical layer (CL) is a thin layer containing the surface $z_2 = z_{2c}$ (the so-called critical level) at which the phase velocity of the perturbation is equal to the flow velocity. This surface is singular for the equation describing the neutral mode (e.g. the Rayleigh or Taylor–Goldstein equations). For a correct description of the perturbation structure inside the CL one should take into account viscous unsteady or nonlinear terms of the Navier–Stokes equations (as well as the so-called inertial terms responsible for the structure of the neutral mode). Each of these three is negligible far from the critical level and becomes of the same order as inertial terms only at (small) specific distance from this level: viscous $l_\nu$, unsteady† $l_t$ or nonlinear $l_N$, respectively, where

$$l_\nu = \nu^{1/3}, \quad l_t = \gamma, \quad l_N = A^{1/(2-\alpha)} \tag{1.2}$$

(near the critical level the stream function of the neutral mode $\psi \sim A(z_2 - z_{2c})^\alpha$; $\alpha \leqslant \frac{1}{2}$ in all flows that we know). The largest of scales (1.2) determines the thickness $l$ and the type of CL (the CL may be viscous, unsteady or nonlinear) and, corresponding to this scale, terms in the Navier–Stokes equations which, together with inertial ones, form the leading-order governing equation that describes the perturbation structure inside the CL. This equation is linear in linear (viscous and unsteady) CL regimes and nonlinear in the nonlinear CL regime. Note that only $l_\nu$ does not vary with variation of $A$, hence the CL regime can change (and does really change!) in the process of evolution (for more details see Churilov & Shukhman 1993).

One should distinguish between the linear/nonlinear CL (see above) and the linear/nonlinear problem of perturbation development or, equivalently, the linear/nonlinear in $A$ evolution equation (1.1).‡ For example, we can consider the nonlinear evolution of a perturbation in one of the linear CL regimes (which is the subject of this paper) and even linear evolution in the nonlinear CL regime (e.g. Maslowe 1973), but in the last case we are in fact dealing with the late stage of nonlinear evolution (Reutov 1982).

Let us consider now the evolution of an initially very small perturbation. At the first stage of development the evolution equation (1.1) is obviously linear, $F(A) \approx \gamma_L A$, amplitude $A$ rises exponentially with linear growth rate $\gamma_L$ ($\gamma_L$ is also considered as a measure of supercriticality), $l_t = \gamma_L$ does not depend on $A$ and the CL regime is linear – viscous (if $\gamma_L < \nu^{1/3}$) or unsteady (if $\gamma_L > \nu^{1/3}$).

The 'two-dimensional' analysis has demonstrated that in the general case the neutral mode is singular at the critical level and for this reason the perturbation magnitude inside the CL§ is much greater than outside it. Because nonlinearity in (1.1) is 'produced' inside the CL, it is relatively strong and has already become competitive (i.e. of the same order as $\gamma_L A$) in the corresponding linear CL regime (see above). Just after becoming competitive, the nonlinearity tries to either decelerate ('stabilizing' nonlinearity) or accelerate ('destabilizing' nonlinearity) the perturbation growth. In

† We extend the terms 'unsteady CL' and 'unsteady scale', introduced initially for problems of temporal evolution, to spatial evolution.

‡ Linearity of (1.1) means that the part of $F(A)$ nonlinear in $A$ is small as compared with the linear one, rather than absence of any nonlinearity. Note that the main nonlinearity in (1.1) is due to the CL (for more extensive discussion see Churilov & Shukhman 1993).

§ The amplitude is a (global) measure of the perturbation as a whole, whereas the magnitude is its local (at given point or in small domain) value.

the viscous CL regime stabilizing nonlinearity saturates the instability, and we have the so-called supercritical Hopf bifurcation, whereas destabilizing nonlinearity leads to explosive growth of the perturbation: $A \sim (\xi_0 - \xi)^{-1/2}$. In this case the unsteady scale $l_t$ rises with $A$, then becomes greater than $l_\nu$, and the viscous CL regime is replaced by the unsteady CL regime. Most surprising is the fact that in the unsteady CL regime both types of nonlinearity – stabilizing as well as destabilizing – lead to explosive evolution:

$$A \sim (\xi_0 - \xi)^{-5/2+\alpha} \tag{1.3}$$

(for discussion see Churilov & Shukhman 1988). Note that amplitude growth in this case is so fast that the unsteady scale $l_t$ remains the largest up to $A \sim O(1)$ where the weakly nonlinear analysis becomes invalid.

There are, however, some flows (e.g. free shear flow of an incompressible non-stratified fluid) in which the neutral mode is regular at the critical level ($\alpha = 0$) and, therefore, the perturbation magnitude inside the CL is not as large. Hence, in such a flow the nonlinearity in the evolution equation (1.1) is much weaker than in flows with singular neutral modes and turns out to be non-competitive in linear CL regimes (at least for $\gamma_L > \nu^{2/3}$). An unstable perturbation in this case rises exponentially up to a certain amplitude (the so-called boundary of the nonlinear CL regime) at which the nonlinear scale $l_N$ becomes of the same order as the corresponding linear one, and then continues to rise slowly in a power-like manner, $A \sim \xi^{2/3}$ (Huerre & Scott 1980; Churilov & Shukhman 1987; Goldstein & Hultgren 1988). For a more extensive discussion of 'two-dimensional' theory see Churilov & Shukhman (1993).

In 1989 Goldstein & Choi considered the spatial evolution of a *three-dimensional* disturbance in the form of a pair of oblique waves of equal amplitude in a free shear flow and have shown that amplitude $A$ increases downstream in an explosive manner, $A \sim (\xi_0 - \xi)^{-3}$, while a two-dimensional (stream-aligned) wave *in the same flow* increases, as we know, much slower, $A \sim \xi^{2/3}$ (see above). Recently the same result for the temporal evolution was obtained by Wu, Lee & Cowley (1993). In view of this, three-dimensional perturbations can play a more important role in shear flow dynamics and now most investigations in this area are devoted to three-dimensional perturbations.

The reason for such a dramatic difference in the behaviour of two- and three-dimensional disturbances at the nonlinear stage is the much stronger nonlinearity of the evolution equation (1.1) due to the singularity of the three-dimensional neutral mode at the critical level, in contrast with two-dimensional one. Namely, $V_\parallel \sim k_\perp^2 A(z_2 - z_{2c})^{-1}$, $V_\perp \sim k k_\perp A(z_2 - z_{2c})^{-1}$, where $V_\parallel, k$ and $V_\perp, k_\perp$ are streamwise and spanwise components of the perturbed velocity and wave vector, respectively; see also comment in parenthesis after formula (1.2). In what follows, we shall refer to this stronger nonlinearity as an oblique nonlinearity.

Let us, however, pay attention to the fact that the question of oblique nonlinearity is not so simple. At the first sight, a shear flow is a highly anisotropic system (two directions, $z_1$ and $z_2$, are fixed by the velocity and its gradient), and the development of every wave with $k_\perp \neq 0$ (or, more generally, depending on spanwise coordinate $z_3$) seems to be described by an evolution equation with oblique nonlinearity due to the singularity of $V_\parallel$ and $V_\perp$ at the critical level. But it is easy to demonstrate that the evolution of a two-dimensional *oblique* disturbance of the form

$$f = f(s, z_2, t), \quad s = z_1 \cos\beta + z_3 \sin\beta \tag{1.4}$$

(in a linear approximation $f = A\phi_f(z_2)\exp(\mathrm{i}qs - \mathrm{i}\omega t)$ with $A$ being a function of $t$ or $s$) in a flow with unperturbed velocity $V_0(z_2)$ is the same as the evolution of a stream-aligned ($k_\perp = 0$) two-dimensional disturbance in the flow $V_0(z_2)\cos\beta$. In other words, we can rotate the coordinate system in the $(z_1, z_3)$-plane through the angle $\beta$ so that the new $z_1$ axis is along the wave vector $\boldsymbol{k}$, and see that the component of velocity normal to the $\boldsymbol{k}$ (while being singular) does not influence other components.† The result obtained (note that it is valid for the complete nonlinear system of Navier–Stokes equations) forms the basis for the Squire theorem, well-known in linear theory; in what follows, when referring to this result and its consequences, we will speak of the Squire theorem, realizing, however, that it is not strictly this theorem.

Therefore, to have an oblique nonlinearity in the evolution equation one should to consider disturbances that are really three-dimensional. In order to reveal the role of the three-dimensionality, Goldstein & Choi (1989), and subsequently Wu *et al.* (1993), considered a disturbance not in the form of a single oblique wave but in the form of a pair of oblique waves of the same amplitude, $f = A\phi_f(z_2)\cos(k_\perp z_3)\exp(\mathrm{i}kz_1 - \mathrm{i}\omega t)$ and, as has already been pointed out above, obtained an explosive growth of amplitude at the nonlinear stage.

In this paper we consider perturbations in the form of a single oblique wave and examine some ways of violating the Squire theorem for obtaining an oblique nonlinearity. It will be shown that there are at least two possibilities. One is due to the streamwise evolution of perturbations. The point is that in this case the amplitude is a function of $z_1$ (not $s$!), and we cannot reduce the spatial structure of a disturbance to a (1.4) type (with dependence on $s$ and $z_2$ only). The streamwise variation of amplitude permits an oblique nonlinearity to manifest itself, and this leads to an explosive growth, though slower than in the case of a pair of waves (cf. with (1.3); $\alpha = 0$; $\xi = z_1$): $A \sim (z_{10} - z_1)^{-5/2}$. Note that in the temporal evolution problem the (oblique) disturbance remains two-dimensional and evolution is slow.

The other possibility is associated with the curvature of the shear layer. We shall consider not a plane shear flow but an axisymmetric jet where the shear layer is cylindrical. Owing to the non-equivalency of the axial and azimuthal directions, it is impossible to totally eliminate the pole-type singularity in the perturbed velocity by a rotation of axes, and the oblique nonlinearity is proportional to the curvature $D = d/R$ of the shear layer (here $d$ is a typical layer width, $R$ is the radius of its localization; the limit $D \to 0$ corresponds to a plane layer). Also in this case, however, even with a finite curvature $D = O(1)$, the nonlinearity for a helical disturbance ($f = A(z)\,\phi_f(r)\exp\{i(kz + m\phi - \omega t)\}$, where $r, \phi, z$ are cylindrical coordinates) is weak compared with the case of a pair of waves with azimuthal numbers $+m$ and $-m$. This is because the main nonlinearity in the evolution equation is 'produced' inside the CL which is a very thin layer and therefore turns out to be 'quasi-planar' even for $D = O(1)$ and hence produces 'attenuated' nonlinearity.

In this paper a study is made of the nonlinear spatial evolution of helical disturbances of an axisymmetric jet, excited by an external source with frequency $\omega$ and azimuthal number $m$. Such a problem was realized, for example, in a series of experiments reported by Cohen & Wygnanski (1987*a*, *b*), who excited the jet by setting a certain frequency and azimuthal asymmetry in the form of a pattern rotating in $\phi$ at the origin of jet outflow (near the nozzle). We wish, however, to consider only some theoretical questions, with no connection with experiment, to demonstrate the way in which the two above-mentioned possibilities of obtaining an oblique nonlinearity are realized in the case of a single-$m$ helical wave.

† Note that the $\boldsymbol{k}$-aligned component of the perturbed velocity in the neutral mode, $V_\parallel \cos\beta + V_\perp \sin\beta$, turns out to be regular at the critical level, whereas $V_\parallel$ and $V_\perp$ are indeed singular.

The problem under consideration involves one more interesting result. Wu *et al.* (1993), by examining the temporal evolution of a pair of oblique waves, showed that in the regime of a viscous CL the evolution equation has a nonlinearity unusual for this CL regime: instead of a local nonlinear term (as in the Landau–Stuart–Watson equation), it contains a non-local (i.e. integral in the evolution variable) nonlinear term with a 'memory'† which in the case of a destabilizing sign leads to an explosive growth of a disturbance, and in the case of a stabilizing sign it leads not simply to saturation but to total damping of the perturbation. It will be shown that for a single oblique (helical) wave the nonlinear term in the evolution equation in the regime of a viscous CL also has an integral form, though somewhat different than in the case of a pair of oblique waves.

The paper is organized as follows. In §2 we will give information about the linear stability theory needed for the subsequent discussion, placing emphasis on the properties of weakly supercritical disturbances. In §3 the derivation of the nonlinear evolution equation (NEE) will be given, and in §4 an analysis will be made of the properties of its solutions. The results obtained will be discussed in §5. The Appendix is devoted to analysing the $m = 0$ mode and to taking into account effects associated with the viscous broadening of an unperturbed flow.

## 2. Linear theory

### 2.1. *The model*

Let us introduce cylindrical coordinates $r, \phi, z$ and consider an axial jet with unperturbed velocity $V_z = w(r)$ which is maximal at the centre and gradually decreases to zero toward the periphery:

$$w(r) = \bar{w}(1-\hat{w}(r)),$$

where $\hat{w}(r)$ is a monotonic function, such that $\hat{w}(r=0) = -1, \hat{w}(r=\infty) = 1$. Quite a number of model profiles for jets have been considered in the literature (see, for example, Cohen & Wygnanski 1987*a* and examples therein). We do not specify $w(r)$, but for illustrative purposes we will give, on occasion, results of calculations for the model

$$w(r) = \bar{w}\left[1-\tanh\left(\frac{1}{D}\ln\frac{r}{R_0}\right)\right]. \tag{2.1}$$

In the limit of small curvature ($D \rightarrow 0$) we obtain, of course a plane shear flow. Our notation is related to that usually used in the plane-critical-layer literature as follows:

$$r-R_0 \rightarrow y, \quad R_0\phi \rightarrow z, \quad z \rightarrow x;$$
$$v_r \rightarrow v, \quad v_\phi \rightarrow w, \quad v_z \rightarrow u.$$

In view of this the model (2.1) in the limit $D \rightarrow 0$ becomes the widely used plane layer model

$$\frac{w(r)}{\bar{w}} = 1-\tanh(y/d), \tag{2.2}$$

where $d = DR_0$ is the shear layer width.

† Note that while evolution equations with a non-local nonlinearity are well known in the unsteady CL regime, Wu *et al.* (1993) and Smith & Blennerhassett (1992) were the first to obtain such an equation in the *viscous* CL regime.

Now we shall justify studying weakly unstable modes or, equivalently, applying the weakly nonlinear theory to a system having a wide (in the $\boldsymbol{k}$-space) spectrum of unstable modes with $\max_k \gamma_L = O(1)$. This difficulty is extensively discussed in the literature and there are some ways to overcome it (e.g. see Goldstein & Hultgren 1988).

We prefer to assume that a perturbation is produced by an external source which sets a proper frequency $\omega$ and azimuthal number $m$. We suppose also that the amplitude of a perturbation generated in such a way is large enough to neglect more unstable disturbances that could arise from (very-low-amplitude) noise and small enough to obey the linear evolution equation near the source. This selected perturbation increases downstream, and at some distance its evolution becomes nonlinear; this is the process we are studying.

The Reynolds number $Re = \bar{w}d/\nu$ is large enough that viscosity only needs to be taken into account inside the CL. The viscous broadening of the jet is assumed to be slow enough and it will be neglected in the main body of the paper, although some effects due to the broadening are considered in §5 and the Appendix.

In this paper we study the spatial nonlinear evolution, i.e. the dependence of the perturbation amplitude on streamwise coordinate $z$. Because we shall do it in the framework of a weakly nonlinear theory, we assume that the disturbance is a weakly supercritical one, i.e. the frequency $\omega$ is slightly less than a critical value $\omega_{cr}$: $|\omega-\omega_{cr}| \ll \omega_{cr}$ where $\omega_{cr}$ depends on curvature $D$. Thus, in the case of a plane flow ($D \to 0$) in the model (2.2) $\omega_{cr} = \bar{w}/d$, which corresponds to the critical Strouhal number $St_{cr} = d\omega_{cr}/\bar{w}$ equal to unity.

### 2.2. *Analysis of a neutral stability*

Linear stability theory of an axisymmetric jet has been studied in sufficient detail (see, for example, Michalke 1965, Cohen & Wygnanski 1987*a*): for a number of models the dependence of growth rate on frequency was calculated at different $m$, eigenfunctions were obtained, etc. Since we are interested in the case of weakly unstable disturbances, it is necessary to know the neutral mode properties as well as the dependence of a critical frequency and the longitudinal wavelength on curvature $D$ at different $m$. Based on this information one can assess, for example, the relative importance of the modes at different $D$.

To solve the problem, we can use either the linear inviscid equation for perturbed pressure

$$p''+p'\left(\frac{1}{r}+\frac{2w'}{c-w}\right)-\left(\frac{m^2}{r^2}+k^2\right)p = 0, \tag{2.3}$$

with the boundary conditions $p(0) = 0$ and $p \to 0$ as $r \to \infty$, or the equation for the disturbance of the radial velocity component

$$v_r''+\frac{1}{r}v_r'-\frac{1}{r^2}v_r+\frac{w''-w'/r}{c-w}v_r-\left(\frac{m^2}{r^2}+k^2\right)v_r+\frac{2m^2}{r^3}\frac{1}{m^2/r^2+k^2}\left(v_r'+\frac{1}{r}v_r+\frac{w'}{c-w}v_r\right) = 0, \tag{2.4}$$

with the boundary conditions $v_r \to 0$ as $r \to \infty$ and $|v_r| < \infty$ as $r \to 0$. Here $c = \omega/k$, and the disturbance is taken in the form $\sim \exp\{\mathrm{i}(m\phi+kz-\omega t)\}$. Equations (2.3) and (2.4) are, generally speaking, singular at $r = r_c$ (where $w = c$), and we need to consider the solution inside the CL which matches the solution of (2.3)/(2.4) at $r = r_c-0$ to one at $r = r_c+0$. In the linear theory there are only two types of CL – viscous and unsteady – and they both provide the same way of extending the solution through $r = r_c$ – Lin's indentation rule: since $w_c' < 0$, the indentation should proceed from above ($\mathrm{Im}(r) > 0$).

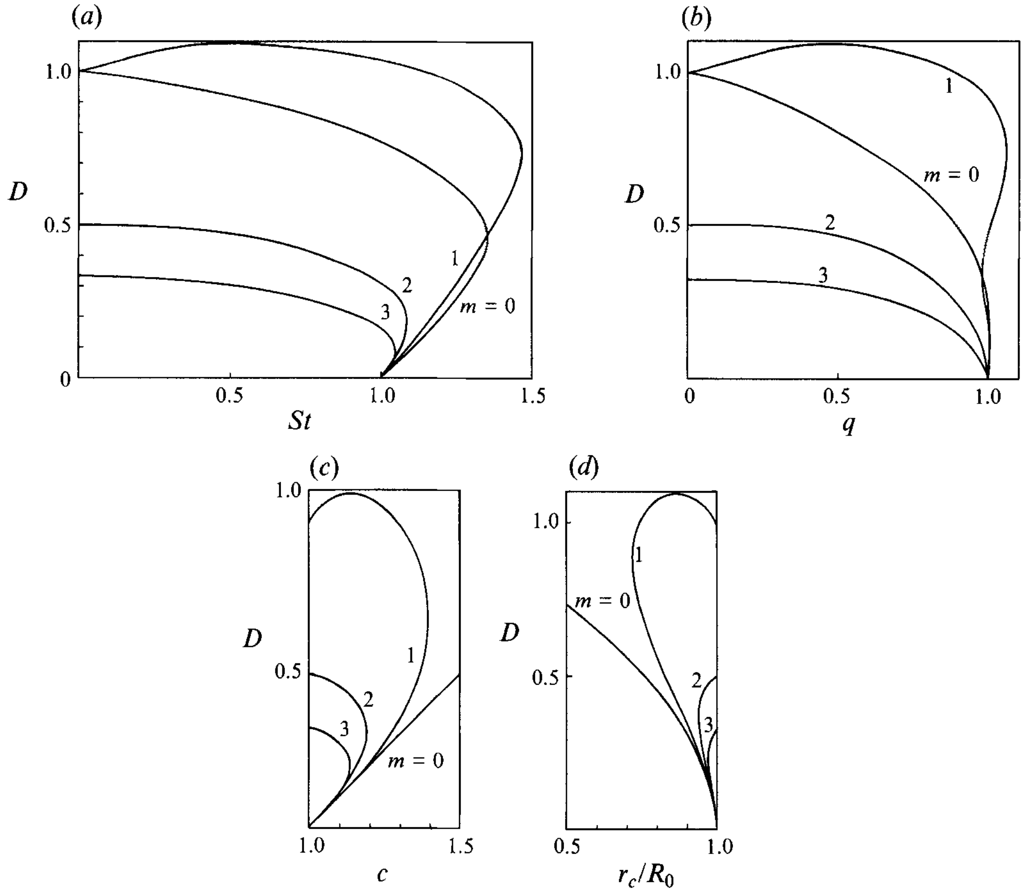


FIGURE 1. Neutral curves of the modes $m = 0, 1, 2, 3$ for the flow model (2.1): the dependence on the curvature parameter $D$ of ($a$) the critical Strouhal number $St = \omega d/\bar{w}$; ($b$) the wavenumber of the neutral mode $q = kd$; ($c$) the velocity $c$ of the neutral mode; ($d$) the position of critical level $r_c$.

It is easy to show that real boundary conditions require that there is no logarithmic contribution to expansions of the eigenfunctions (2.3) and (2.4) on the CL (see (2.15)), which imposes the condition

$$\left(\frac{w_c''}{w_c'}-\frac{1}{r_c}\right)\left(\frac{m^2}{r_c^2}+k^2\right)+\frac{2m^2}{r_c^3}=0. \tag{2.5}$$

Figures 1($a$)–1($d$) show the results of calculations of neutral curves using the model (2.1) as well as phase velocities $c$ and position $r_c$ for four modes: $m = 0, 1, 2, 3$. The qualitative properties of these curves appear to be the same for other similar jet velocity profiles.

Figure 1($a$) gives the $D$-dependences of the critical Strouhal number $St = St(D)$, $St = \omega d/\bar{w}, d = R_0 D$; the instability region of mode $m$ lies under the respective curve. Let us emphasize some properties of the neutral curves:

(i) In the case of a plane layer the critical Strouhal numbers for all modes coincide: $St_m = 1$ (see also Cohen & Wygnanski 1987$a$).

(ii) With increasing parameter of curvature $D$, the region of unstable values of $St$ expands initially and then starts to decrease, becoming zero.

(iii) For each of the modes $m$, there exists a maximum value of the parameter $D$ at which it can still be unstable. In the model (2.1) modes with $m > 2$ are stabilized when $D > 1/m$.

(iv) At small $D$ the widest instability region corresponds to the mode $m = 0$; however, with increasing $D$, the mode $m = 1$ becomes dominant (i.e. its growth rate becomes the largest one).

(v) For all helical modes ($m \neq 0$) we have $r_c \sim R_0$, i.e. the CL is located in the middle part of the shear layer.

Let us also give the results of analytic calculation of the neutral curves for the model (2.1). On the right-hand edge (at small $D$):

$$St_m = 1 + D - \tfrac{1}{2}\alpha_m D^2, \quad q = 1 - \tfrac{1}{2}\alpha_m D^2. \tag{2.6}$$

Here $$q = kd, \quad \alpha_m = m^2 + 1 + \pi^2/6 - J = m^2 - 0.2020,$$

where $$J = \int_{-\infty}^{\infty} \frac{z^2 \ln(2\cosh z)}{\cosh^2 z} \mathrm{d}z = 2.8470.$$

One the left-hand edge ($St \ll 1, q \ll 1$):
for $m \geqslant 2$

$$St = q(1 + Dq^2), \quad (mD)^2 = 1 - q^2 \pi D \cot(\pi D); \tag{2.7}$$

for $m = 1$

$$St = q(1 + q^2), \quad D = 1 + q^2\left(\ln\frac{2}{q} - C - \frac{1}{2}\right); \tag{2.8}$$

and for $m = 0$

$$St = q(1 + D), \quad D = 1 - q^2\left(\ln\frac{2}{q} - C - \frac{1}{2}\right). \tag{2.9}$$

Here $C = 0.577216$ is the Euler constant.

### 2.3. *Instability*

By fixing $D$ and slightly departing from the neutral curve towards the instability region, i.e. assuming $\omega = \omega_{cr} - \omega_1$ ($\omega_1 > 0$ always, except for $m = 1$ when $D > 1$, see figure 1*a*) and $k \to k + k_1$ and using the familiar procedure of perturbation theory, from (2.3) we obtain

$$k_1 J_2 + \omega_1 J_1 = 0, \tag{2.10}$$

where $$J_1 = \frac{2}{k}\int_0^{\infty} \frac{w' \phi_a \phi_a'}{(c-w)^4} r\,\mathrm{d}r, \quad J_2 = cJ_1 - 2k\int_0^{\infty} \frac{\phi_a^2}{(c-w)^2} r\,\mathrm{d}r. \tag{2.11}$$

The singularity in the integrals (2.11) is indented from above, and $\phi_a(r)$ is an eigenfunction of the neutral mode of pressure, calibrated by the condition $\phi_a(r_c) = 1$. Physically this means that we have chosen the amplitude of the disturbed pressure as an amplitude of perturbation $A$. The dispersion equation (2.10) can also be written as a linear evolution equation. Assuming $k_1 A \to -\mathrm{i}\mathrm{d}A/\mathrm{d}z$ we get

$$\frac{\mathrm{d}A}{\mathrm{d}z} + \mathrm{i}\omega_1(J_1/J_2)A = 0. \tag{2.12}$$

The spatial growth rate of a disturbance is

$$\gamma_L = -\operatorname{Im}(k_1) = \omega_1 \operatorname{Im}(J_1/J_2) \tag{2.13}$$

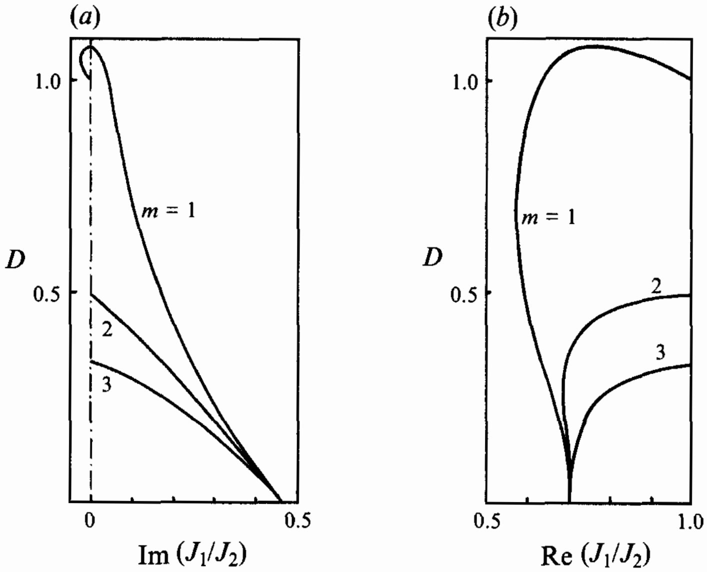


FIGURE 2. The dependene of (*a*) $\mathrm{Im}(J_1/J_2)$ and (*b*) $\mathrm{Re}(J_1/J_2)$ on $D$ for $m = 1, 2, 3$.

and a correction to the streamwise wavenumber is

$$\Delta k_r = -\omega_1 \,\mathrm{Re}(J_1/J_2). \tag{2.14}$$

The quantities $\mathrm{Im}(J_1/J_2)$ and $\mathrm{Re}(J_1/J_2)$ are given in figures 2(*a*) and 2(*b*) respectively, as functions of $D$ for the model (2.1).

To conclude this Section, we give the Frobenius expansion of the eigenfunction of the neutral mode of the pressure near the CL for the general case:

$$\phi_a = 1 + \beta\xi^2 + \delta\xi^4 + (\kappa + \sigma \ln|\xi|)(\xi^3 + \alpha\xi^4 + \ldots) + \ldots. \tag{2.15}$$

Here

Here

$$\left.\begin{aligned} \alpha &= \frac{3}{4}\left(\frac{w_c''}{w_c'} - \frac{1}{r_c}\right); \quad \beta = -\frac{1}{2}\left(\frac{m^2}{r_c^2} + k^2\right); \\ 4\delta &= \beta\left(k^2 + \frac{m^2}{r_c^2}\right) + 3\frac{m^2}{r_c^4} + 2\beta\left(\frac{1}{r_c^2} + \frac{2}{3}\frac{w_c'''}{w_c'} - \frac{1}{2}\left(\frac{w_c''}{w_c'}\right)^2\right); \\ \sigma &= -\frac{1}{6}\left\{\left(\frac{w_c''}{w_c'} - \frac{1}{r_c}\right)\left(k^2 + \frac{m^2}{r_c^2}\right) + \frac{2m^2}{r_c^3}\right\}, \quad \xi = r - r_c, \end{aligned}\right\} \tag{2.16}$$

and the coefficient $\kappa$ can be determined only after solving the boundary-value problem (we do not need its explicit value). As has been pointed out above, the boundary conditions require $\sigma = 0$ on the CL (cf. (2.5)).

## 3. Derivation of the nonlinear evolution equation

### 3.1. *Scaling*

Let us make all quantities dimensionless so that the shear layer width $d = 1$ and the typical mean flow velocity $\bar{w} = 1$. The dimensionless viscosity is the inverse Reynolds number, and the dimensionless frequency $\omega$ is the Strouhal number. Let $\epsilon$ and $\mu$ be small parameters, characterizing the disturbance amplitude, $A = \epsilon\tilde{A}$ (subsequently, the tilde is omitted), and supercriticality, $\omega_1 = \mu\Omega$. To cover simultaneously the cases of both unsteady and viscous CL regimes, we put $\nu = \mu^3\eta$; in this case $l_\nu \sim l_t$ (see (1.2)). The relationship between $\mu$ and $\epsilon$ is established in the course of subsequent iterations. It is found that

$$\epsilon = \mu^{5/2}. \tag{3.1}$$

We introduce a 'slow' (evolution) coordinate $\zeta = \mu z$ and assume further that the dependence of all functions on coordinates and time has the form

$$F = F(\zeta, Z; r), \quad \text{where} \quad Z = z + \frac{m}{k}\phi - ct + \mu t\frac{\Omega}{k}.$$

The procedure for deriving the NEE has been reproduced in numerous publications; therefore, here we give only a brief outline.

### 3.2. *The outer problem*

We omit the details of the derivation and give the inner asymptotic expansion of the outer solution. In order to obtain it, it is sufficient to have equation (2.3) and relationships relating the velocity components $v_r, v_\phi$ and $v_z$ to the perturbed pressure:

$$v_{r1} = -\frac{\mathrm{i}}{k}\frac{p_1'}{c-w}, \quad v_{\phi 1} = \frac{m}{kr}\frac{p_1}{c-w}, \quad v_{z1} = \frac{p_1}{c-w} - \frac{w'}{k^2}\frac{p_1'}{(c-w)^2}. \tag{3.2}$$

Here $v_{r1}, v_{\phi 1}, v_{z1}, p_1$ are the fundamental harmonics ($n = 1$) of the velocity and pressure perturbations:

$$f = \sum_{n=-\infty}^{\infty} f_n(r, \zeta)\exp(\mathrm{i}nkZ), \quad \bar{f}_{-n} = f_n.$$

The overbar denotes complex conjugacy. Assuming $\omega \to \omega - \mu\Omega$, $k \to k - \mathrm{i}\mu\partial/\partial\zeta$ and $r - r_c = \mu Y$, from (3.2) and (2.3) we obtain

$$\left.\begin{aligned} p_1 &= \epsilon(p_1^{(0)} + \mu p_1^{(1)} + \mu^2 p_1^{(2)} + \mu^3 p_1^{(3)} + \mu^4 p_1^{(4)} + \ldots), \\ v_{r1} &= \epsilon(v_{r1}^{(0)} + \mu v_{r1}^{(1)} + \mu^2 v_{r1}^{(2)} + \ldots), \\ v_{\phi 1} &= \epsilon(\mu^{-1} v_{\phi 1}^{(-1)} + v_{\phi 1}^{(0)} + \ldots), \quad v_{z1} = \epsilon(\mu^{-1} v_{z1}^{(-1)} + v_{z1}^{(0)} + \ldots), \end{aligned}\right\} \tag{3.3}$$

where

$$\left.\begin{aligned} &p_1^{(0)} = A, \quad p_1^{(1)} = 0, \quad p_1^{(2)} = A\beta Y^2 + C_1 Y, \quad p_1^{(3)} = A\kappa Y^3 + C_2 Y^2, \\ &\qquad p_1^{(4)} = A(\delta + \alpha\kappa) Y^4 + (C_3^{\pm} + C_4 \ln|\mu Y|) Y^3; \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} v_{r1}^{(0)} &= \frac{2\mathrm{i}\beta}{kw_c'} A, \\ v_{r1}^{(1)} &= \frac{1}{k^2 w_c'}\left(\frac{m^2}{r_c^2} - k^2\right)\frac{\mathrm{d}A}{\mathrm{d}\zeta} + \frac{1}{(kw_c')^2}\left(3\kappa - \frac{w_c''}{w_c'}\beta\right)\hat{\mathscr{L}}_1 A + \frac{2\beta}{(kw_c')^2}\frac{1}{r_c}\hat{l}A + O(Y^{-2}), \\ v_{r1}^{(2)} &= \frac{\mathrm{i}}{kw_c'}(T + 3C_3^{\pm}) Y + \frac{3\mathrm{i}}{kw_c'} C_4 Y \ln|\mu Y| \\ &\quad + \frac{\mathrm{i}}{kw_c'} Y^2 \left\{4(\delta + \alpha\kappa) - \tfrac{3}{2}\kappa\frac{w_c''}{w_c'} + 2\beta\left[\left(\frac{w_c''}{2w_c'}\right)^2 - \frac{w_c'''}{6w_c'}\right]\right\}, \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} v_{\phi 1}^{(-1)} &= -\frac{1}{kw_c'}\frac{m}{r_c}AY^{-1} - \frac{im}{r_c}\frac{1}{(kw_c')^2}Y^{-2}\hat{l}A, \\ v_{\phi 1}^{(0)} &= \frac{1}{kw_c'}\frac{m}{r_c}\left(\frac{1}{r_c}+\frac{w_c''}{2w_c'}\right)A + O(Y^{-1}), \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} v_{z1}^{(-1)} &= \frac{1}{k^2 w_c'}\frac{m^2}{r_c^2}AY^{-1} + \frac{\mathrm{i}}{k(kw_c')^2}\frac{m^2}{r_c^2}Y^{-2}\hat{l}A, \\ v_{z1}^{(0)} &= \frac{1}{k^2 w_c'}A\left(\frac{w_c''}{2w_c'}k^2 - 3\kappa\right) + O(Y^{-1}). \end{aligned}\right\} \tag{3.7}$$

Here we use the notation

$$\left.\begin{aligned} \hat{l} &= c\frac{\partial}{\partial \zeta} + \mathrm{i}\Omega, \\ \hat{\mathscr{L}}_n &= c\frac{\partial}{\partial \zeta} + \mathrm{i}n(kw_c' Y + \Omega) - \eta\frac{\partial^2}{\partial Y^2}, \end{aligned}\right\} \tag{3.8}$$

$$\left.\begin{aligned} C_1 &= -\frac{2\mathrm{i}\beta}{kw_c'}\hat{l}A, \quad C_2 = \mathrm{i}k\frac{\mathrm{d}A}{\mathrm{d}\zeta} + \mathrm{i}\left(\frac{m^2}{r_c^3} - 3\kappa\right)\frac{1}{rw_c'}\hat{l}A, \\ 3C_4 &= \frac{2\mathrm{i}}{kw_c'}\left(\frac{m^2}{r_c^4} + \frac{\beta w_c'''}{w_c'}\right)\hat{l}A + 2\mathrm{i}k\left(\frac{w_c''}{w_c'} - \frac{1}{r_c}\right)\frac{\mathrm{d}A}{\mathrm{d}\zeta}. \end{aligned}\right\} \tag{3.9}$$

$A = A(\zeta)$ is the dimensionless amplitude of the neutral mode of the pressure, $p_1 = A\phi_a(r)$ (see (2.15)), and $T$ is a coefficient that has no jump on the CL. Generally speaking, expansions of the zeroth and second harmonics should be added to the above expansions, but since their matching with the asymptotic expansion of the inner solution proceeds automatically and does not provide any new information, we omit them.

### 3.3. *The modified solvability condition*

Now we complement (3.9) with a relationship for the coefficients $C_3^{\pm}$ that are also involved in the expansions (3.4) and (3.5):

$$\mathrm{i}J_2\left(\frac{\mathrm{d}A}{\mathrm{d}\zeta} + \mathrm{i}\Omega\frac{J_1}{J_2}A\right) = -\frac{3r_c}{w_c'^2}(C_3^+ - C_3^- + \mathrm{i}\pi C_4), \tag{3.10}$$

which is called the modified solvability condition (MSC). By calculating the coefficients $C_3^{\pm}$ from the inner solution and substituting into (3.10), we obtain the evolution equation.

In a linear approximation, when the CL is viscous or unsteady, the result is known: $C_3^+ - C_3^- = -\mathrm{i}\pi C_4$, and we, of course, obtain the linear evolution equation (2.12) derived previously by means of the Lin indentation rule. Nonlinearity makes an additional contribution:

$$C_3^+ - C_3^- = -\mathrm{i}\pi C_4 + (C_3^+ - C_3^-)_N, \tag{3.11}$$

such that the desired nonlinear evolution equation takes the form

$$\mathrm{i}J_2\left(\frac{\mathrm{d}A}{\mathrm{d}\zeta} + \mathrm{i}\Omega\frac{J_1}{J_2}A\right) = -\frac{3r_c}{w_c'^2}(C_3^+ - C_3^-)_N. \tag{3.12}$$

### 3.4. *The inner problem*

By denoting $v_r = \epsilon V, v_\phi = \epsilon U, v_z = w(r)+\epsilon W$, and $p = p_0+\epsilon P$, we write the complete system for nonlinear equations:

$$\left(\frac{\partial}{\partial t}+w\frac{\partial}{\partial z}\right)V+\frac{\partial P}{\partial r}-\nu\left(\Delta V-\frac{V}{r^2}-\frac{2}{r^2}\frac{\partial U}{\partial \phi}\right) = -\epsilon\left(V\frac{\partial}{\partial r}+\frac{U}{r}\frac{\partial}{\partial \phi}+W\frac{\partial}{\partial z}\right)V+\epsilon\frac{U^2}{r}, \quad (3.13)$$

$$\left(\frac{\partial}{\partial t}+w\frac{\partial}{\partial z}\right)U+\frac{1}{r}\frac{\partial P}{\partial \phi}-\nu\left(\Delta U-\frac{U}{r^2}+\frac{2}{r^2}\frac{\partial V}{\partial \phi}\right) = -\epsilon\left(V\frac{\partial}{\partial r}+\frac{U}{r}\frac{\partial}{\partial \phi}+W\frac{\partial}{\partial z}\right)U-\epsilon\frac{UV}{r}, \quad (3.14)$$

$$\left(\frac{\partial}{\partial t}+w\frac{\partial}{\partial z}\right)W+Vw'+\frac{\partial P}{\partial z}-\nu\Delta W = -\epsilon\left(V\frac{\partial}{\partial r}+\frac{U}{r}\frac{\partial}{\partial \phi}+W\frac{\partial}{\partial z}\right)W, \quad (3.15)$$

$$\frac{\partial V}{\partial r}+\frac{V}{r}+\frac{1}{r}\frac{\partial U}{\partial \phi}+\frac{\partial W}{\partial z} = 0, \quad (3.16)$$

where $\Delta = r^{-1}(\partial/\partial r)(r\partial/\partial r)+r^{-2}\partial^2/\partial\phi^2+\partial^2/\partial z^2$ is the Laplace operator.

In view of the scaling introduced in §3.1, we rewrite (3.13)–(3.16) by using the inner variable $Y$ $(r-r_c = \mu Y)$ and retaining only terms needed subsequently:

$$\left(\mu\frac{\Omega}{k}-c\right)\frac{\partial V}{\partial z}+\left(c+w_c'\mu Y+\tfrac{1}{2}w_c''\mu^2Y^2+\tfrac{1}{6}w_c'''\mu^3Y^3\right)\left(\frac{\partial V}{\partial z}+\mu\frac{\partial V}{\partial \zeta}\right)+\mu^{-1}P_Y$$
$$-\eta\mu^3\left\{\mu^{-2}V_{YY}+\frac{1}{r_c}\mu^{-1}V_Y-\frac{2}{r_c^2}\frac{\partial U}{\partial \phi}+\frac{\partial^2 V}{\partial z^2}+\frac{1}{r_c^2}\frac{\partial^2 V}{\partial \phi^2}\right\}$$
$$= -\epsilon\left[\mu^{-1}VV_Y+\frac{U}{r_c}\left(1-\mu\frac{Y}{r_c}\right)\frac{\partial V}{\partial \phi}+W\left(\frac{\partial V}{\partial z}+\mu\frac{\partial V}{\partial \zeta}\right)\right]+\epsilon\frac{U^2}{r_c}, \quad (3.17a)$$

$$\left(\mu\frac{\Omega}{k}-c\right)\frac{\partial U}{\partial z}+\left(c+w_c'\mu Y+\tfrac{1}{2}w_c''\mu^2Y^2+\tfrac{1}{6}w_c'''\mu^3Y^3\right)\left(\frac{\partial U}{\partial z}+\mu\frac{\partial U}{\partial \zeta}\right)+\frac{1}{r_c}\frac{\partial P}{\partial \phi}\left(1-\mu\frac{Y}{r_c}\right)$$
$$-\eta\mu^3\left\{\mu^{-2}U_{YY}+\frac{1}{r_c}\mu^{-1}U_Y\left(1-\mu\frac{Y}{r_c}\right)+\frac{\partial^2 U}{\partial z^2}+\frac{1}{r_c^2}\frac{\partial^2 U}{\partial \phi^2}+\frac{2}{r_c^2}\frac{\partial V}{\partial \phi}\right\}$$
$$= -\epsilon\left[\mu^{-1}VU_Y+\frac{U}{r_c}\left(1-\mu\frac{Y}{r_c}+\mu^2\frac{Y^2}{r_c^2}\right)\frac{\partial U}{\partial \phi}+W\left(\frac{\partial U}{\partial z}+\mu\frac{\partial U}{\partial \zeta}\right)\right]-\epsilon\frac{UV}{r_c}\left(1-\mu\frac{Y}{r_c}\right), \quad (3.17b)$$

$$\left(\mu\frac{\Omega}{k}-c\right)\frac{\partial W}{\partial z}+(c+w_c'\mu Y+\tfrac{1}{2}w_c''\mu^2Y^2+\tfrac{1}{6}w_c'''\mu^3Y^3)\left(\frac{\partial W}{\partial z}+\mu\frac{\partial W}{\partial \zeta}\right)+\frac{\partial P}{\partial z}+\mu\frac{\partial P}{\partial \zeta}$$
$$-\eta\mu^3\left\{\mu^{-2}W_{YY}+\frac{1}{r_c}\mu^{-1}W_Y\left(1-\mu\frac{Y}{r_c}\right)+\frac{\partial^2 W}{\partial z^2}+\frac{1}{r_c^2}\frac{\partial^2 W}{\partial \phi^2}\right\}+V\left(w_c'+\mu w_c''Y+\frac{w_c'''}{2}\mu^2Y^2\right)$$
$$= -\epsilon\left[\mu^{-1}VW_Y+\frac{U}{r_c}\left(1-\mu\frac{Y}{r_c}\right)\frac{\partial W}{\partial \phi}+W\left(\frac{\partial W}{\partial z}+\mu\frac{\partial W}{\partial \zeta}\right)\right], \quad (3.17c)$$

$$\mu^{-1}V_Y+\frac{V}{r_c}\left(1-\mu\frac{Y}{r_c}\right)+\frac{1}{r_c}\frac{\partial U}{\partial \phi}\left(1-\mu\frac{Y}{r_c}+\mu^2\frac{Y^2}{r_c^2}\right)+\frac{\partial W}{\partial z}+\mu\frac{\partial W}{\partial \zeta} = 0 \quad (3.17d)$$

As done in the outer problem, we represent all quantities as an expansion in terms of harmonics

$$F = \{F_1(\zeta, Y)\exp(\mathrm{i}Z)+F_2(\zeta, Y)\exp(2\mathrm{i}Z)+\text{c.c.}\}+F_0(\zeta, Y) \quad (3.18)$$

and, in accordance with (3.3), we will seek each of the harmonics in the form of an expansion in a power series of $\mu$:

$$\left.\begin{aligned} P_1 &= P_1^{(0)}+\mu P_1^{(1)}+\mu^2 P_1^{(2)}+\mu^3 P_1^{(3)}+\mu^4 P_1^{(4)}+\ldots, \\ V_1 &= V_1^{(0)}+\mu V_1^{(1)}+\mu^2 V_1^{(2)}+\ldots, \\ U_1 &= \mu^{-1} U_1^{(-1)}+U_1^{(0)}+\mu U_1^{(1)}+\ldots, \\ W_1 &= \mu^{-1} W_1^{(-1)}+W_1^{(0)}+\mu W_1^{(1)}+\ldots; \end{aligned}\right\} \tag{3.19}$$

the zeroth harmonic

$$\left.\begin{aligned} V_0 &= \epsilon(\mu^{-1} V_0^{(-1)}+\ldots), \\ U_0 &= \epsilon(\mu^{-3} U_0^{(-3)}+\mu^{-2} U_0^{(-2)}+\ldots), \\ W_0 &= \epsilon(\mu^{-3} W_0^{(-3)}+\mu^{-2} W_0^{(-2)}+\ldots); \end{aligned}\right\} \tag{3.20}$$

the second harmonic

$$U_2 = \epsilon(\mu^{-3} U_2^{(-3)}+\ldots). \tag{3.21}$$

We have written only those terms in the expansion which are important for subsequent calculations. The jump $C_3^+ - C_3^-$, appearing in the MSC (3.10) is involved at $O(\epsilon\mu^2)$ of the expansions (3.4) and (3.5), and the main nonlinear contribution to it from the inner solution is $O(\epsilon^3\mu^{-3})$, and this gives the scaling (3.1): $\epsilon = \mu^{5/2}$. In the problem of a pair of waves the nonlinearity is found to be stronger, $\sim O(\epsilon^3\mu^{-4})$, which leads to the scaling $\epsilon = O(\mu^3)$ (Goldstein & Choi 1989; Wu *et al.* 1993).

Subsequent calculations have a rather routine character; however, there is one new factor associated with the contribution of the second harmonic. In plane unstratified flows the second harmonic does not usually make contribution to the main nonlinearity (see, for example, Goldstein & Leib 1989; Shukhman 1991), while in a stratified flow, say, the second harmonic makes a contribution of the same order as that of the zeroth harmonic (Churilov & Shukhman 1988). In our problem the nonlinearity caused by the zeroth harmonic is attenuated by an 'incomplete violation' of the Squire theorem whereas the relative role of the second harmonic is enhanced by the flow curvature, and for these reasons they both contribute to the leading order of nonlinearity. Note that in a plane limit this contribution of the second harmonic vanishes. Let us outline the results of consecutive iterations.

3.4.1. *The fundamental harmonic:* $P_1^{(0)}$

From (3.17*a*) at $O(\mu^{-1})$ we have

$$P_{1Y}^{(0)} = 0,$$

whence, in view of the matching to (3.4),

$$P_1^{(0)} = A(\zeta). \tag{3.22}$$

3.4.2. *The fundamental harmonic:* $P_1^{(1)}, U_1^{(-1)}, W_1^{(-1)}, V_1^{(0)}$

From (3.17*a*) at $O(1)$ we have $P_{1Y}^{(1)} = 0$, and matching to the outer solution gives

$$P_1^{(1)} = 0.$$

From (3.17*b, c*) at $O(1)$ and from (3.17*d*) at $O(\mu^{-1})$ we have

$$\hat{\mathscr{L}}_1 U_1^{(-1)} = -\mathrm{i}k_\phi P_1^{(0)}, \tag{3.23}$$

$$\hat{\mathscr{L}}_1 W_1^{(-1)} + V_1^{(0)} w_c' = -\mathrm{i}k P_1^{(0)}, \tag{3.24}$$

$$V_{1Y}^{(0)} + \mathrm{i}(k_\phi U_1^{(-1)} + k W_1^{(-1)}) = 0. \tag{3.25}$$

Here $k_\phi = m/r_c$, and $\hat{\mathscr{L}}_n$ is defined by (3.8). The procedure to be followed below will also be repeated in the subsequent iterations. We multiply (3.23) by $k_\phi$ and (3.24) by $k$, add them together, and substitute the result into (3.25). This yields

$$\hat{\mathscr{L}}_1 V^{(0)}_{1Y} - \mathrm{i}kw'_c V^{(0)}_1 = -(k^2_\phi + k^2) P^{(0)}_1. \tag{3.26}$$

The solution matching to (3.5) is

$$V^{(0)}_1 = -\frac{\mathrm{i}}{kw'_c}(k^2_\phi + k^2) A \equiv \frac{2\mathrm{i}\beta}{kw'_c} A \tag{3.27}$$

and from (3.25) we find that

$$k_\phi U^{(-1)}_1 + k W^{(-1)}_1 = 0. \tag{3.28}$$

Note that at this order the cylindricity has not yet manifested itself, and therefore the Squire theorem is valid, which precisely implies the absence of a term with asymptotic behaviour $Y^{-1}$ as $|Y| \to \infty$ in the velocity projection onto the wave vector (see (3.28)).

From (3.23) and (3.24) we have

$$U^{(-1)}_1 = -\mathrm{i}k_\phi \hat{\mathscr{L}}^{-1}_1 A(\zeta), \quad W^{(-1)}_1 = -\frac{k_\phi}{k} U^{(-1)}_1.$$

Here $\hat{\mathscr{L}}^{-1}_n$ is an operator that is the inverse of $\hat{\mathscr{L}}_n$. Let us give its explicit expression. If $F(\zeta, Y)$ satisfies the equation

$$\hat{\mathscr{L}}_n F(\zeta, Y) = R(\zeta, Y),$$

does not increase exponentially as $Y \to \pm\infty$ and tends to zero as $\zeta \to -\infty$, then

$$F(\zeta, Y) = \hat{\mathscr{L}}^{-1}_n R(\zeta, Y),$$

or in an explicit form

$$F(\zeta, Y) = \int_0^\infty \frac{\mathrm{d}x_1}{c} \int_{-\infty}^\infty \mathrm{d}Y_1 \left(\frac{c}{4\pi\eta x_1}\right)^{1/2} \exp\left[-\frac{c}{4\eta x_1}(Y - Y_1 - \mathrm{i}\eta n k w'_c x_1^2/c^2)^2\right]$$
$$\times \exp\left\{-\frac{\eta}{3}(nkw'_c)^2\left(\frac{x_1}{c}\right)^3 - \mathrm{i}\frac{nx_1}{c}(\Omega + kw'_c Y)\right\} R(\zeta - x_1, Y_1) \tag{3.29}$$

to yield

$$U^{(-1)}_1(\zeta, Y) = -\mathrm{i}k_\phi \int_0^\infty \frac{\mathrm{d}x_1}{c} \exp\left\{-\frac{\eta}{3}(kw'_c)^2\left(\frac{x_1}{c}\right)^3 - \frac{\mathrm{i}x_1}{c}(\Omega + kw'_c Y)\right\} A(\zeta - x_1). \tag{3.30}$$

The asymptotic representation (3.30) as $Y \to \infty$ is matched to (3.6).

3.4.3. *The fundamental harmonic:* $P^{(2)}_1, U^{(0)}_1, W^{(0)}_1, V^{(1)}_1$

From (3.17*a*) at $O(\mu)$ we find

$$P^{(2)}_{1Y} = \frac{\mathrm{i}}{kw'_c}(k^2_\phi + k^2)\hat{l}A + \frac{2}{r_c}(U^{(-3)}_0 U^{(-1)}_1 + U^{(-3)}_2 \overline{U^{(-1)}_1}). \tag{3.31}$$

We write (3.31) as

$$P^{(2)}_{1Y} = \frac{\mathrm{i}}{kw'_c}(k^2_\phi + k^2)\hat{l}A + (P^{(2)}_{1N})_Y. \tag{3.32}$$

Note that a nonlinear contribution (from the centrifugal term $\sim U^2/r$ in (3.17*a*)) is already present at this order.

From (3.17*b*, *c*) at $O(\mu)$ and (3.17*d*) at $O(1)$ we find

$$\hat{\mathscr{L}}_1 U_1^{(0)} + w_c' Y \frac{\partial}{\partial \zeta} U_1^{(-1)} + \tfrac{1}{2} k w_c'' Y^2 U_1^{(-1)} - \frac{ik_\phi}{r_c} Y P_1^{(0)} = \frac{\eta}{r_c} U_{1Y}^{(-1)} - V_1^{(0)} U_{0Y}^{(-3)}, \quad (3.33)$$

$$\hat{\mathscr{L}}_1 W_1^{(0)} + w_c' Y \frac{\partial}{\partial \zeta} W_1^{(-1)} + \tfrac{1}{2} k w_c'' Y^2 W_1^{(-1)} + w_c' V_1^{(1)} + w_c'' Y V_1^{(0)} + ik \frac{dA}{d\zeta} = \frac{\eta}{r_c} W_{1Y}^{(-1)} - V_1^{(0)} W_{0Y}^{(-3)}, \quad (3.34)$$

$$V_{1Y}^{(1)} + \frac{1}{r_c} V_1^{(0)} + ik_\phi \left( U_1^{(0)} - \frac{Y}{r_c} U_1^{(-1)} \right) + ik W_1^{(0)} + \frac{\partial}{\partial \zeta} W_1^{(-1)} = 0. \quad (3.35)$$

These equations give $U_1^{(0)}$, $W_1^{(0)}$ and $V_1^{(1)}$. Note that nonlinear contributions are already contained in $U_1^{(0)}$ and $W_1^{(0)}$; however, by virtue of the relationship for the zeroth harmonics $U_0^{(-3)}$ and $W_0^{(-3)}$, analogous to (3.28) (which will be obtained below),

$$k_\phi U_0^{(-3)} + k W_0^{(-3)} = 0, \quad (3.36)$$

it turns out that $V_1^{(1)}$ does not yet contain nonlinear contributions. At this order the non-zero projection of velocity onto the wave vector has already appeared:

$$k_\phi U_1^{(0)} + k W_1^{(0)} = i \left( V_{iY}^{(1)} + \frac{1}{r_c} V_1^{(0)} \right) + \frac{k_\phi}{r_c} Y U_1^{(-1)} + i \frac{\partial}{\partial \zeta} W_1^{(-1)}. \quad (3.37)$$

Note that this projection is singular in terms of the outer solution (i.e. its asymptotic expansion contains a term $\sim Y^{-1}$ as $|Y| \to \infty$) by virtue of the last term in (3.37). Its origin is unassociated with taking the curvature into account but is wholly due to the spatial statement of the problem: it is not present in the temporal problem.

By repeating for (3.33)–(3.35) the same procedure as in §3.4.2, we obtain the equation for $V_1^{(1)}$:

$$\hat{\mathscr{L}}_1 V_{1Y}^{(1)} - ik w_c' V_1^{(1)} = -\frac{2i\beta}{k w_c'} \frac{1}{r_c} \hat{l} A - \frac{i}{k} (k_\phi^2 - k^2) \frac{dA}{d\zeta} - 2\eta \frac{ik_\phi}{r_c} U_{1Y}^{(-1)},$$

and by differentiating it, we get

$$\hat{\mathscr{L}}_1 V_{1YY}^{(1)} = -2\eta \frac{ik_\phi}{r_c} U_{1YY}^{(-1)}. \quad (3.38)$$

From (3.30) and (3.38), with the help of (3.29) we find

$$V_{1YY}^{(1)} = \frac{2}{r_c} k_\phi^2 \eta (k w_c')^2 \int_0^\infty \frac{dx_1}{c} \int_0^\infty \frac{dx_2}{c} \left( \frac{x_1}{c} \right)^2 \exp \left\{ -\frac{\eta}{3} (k w_c')^2 \left[ \left( \frac{x_1}{c} \right)^3 + 3 \frac{x_1}{c} \frac{x_2}{c} \frac{x_1 + x_2}{c} + \left( \frac{x_2}{c} \right)^3 \right] \right\}$$
$$\times \exp \left\{ -i \frac{x_1 + x_2}{c} (\Omega + k w_c' Y) \right\} A(\zeta - x_1 - x_2). \quad (3.39)$$

For $U_1^{(0)}$ and $W_1^{(0)}$ we write

$$U_1^{(0)} = U_{1L}^{(0)} + U_{1N}^{(0)}, \quad W_1^{(0)} = W_{1L}^{(0)} + W_{1N}^{(0)}. \quad (3.40)$$

We need only the asymptotic representations of $U_{1L}^{(0)}$ and $W_{1L}^{(0)}$ for matching to corresponding expansion terms in (3.6) and (3.7), and the nonlinear contributions $U_N$ and $W_N$. From (3.33), (3.34) and (3.39) one can see that matching is done automatically, and for $U_{1N}^{(0)}$ and $W_{1N}^{(0)}$ we have

$$\hat{\mathscr{L}}_1 U_{1N}^{(0)} = -V_1^{(0)} U_{0Y}^{(-3)}, \quad \hat{\mathscr{L}}_1 W_{1N}^{(0)} = -V_1^{(0)} W_{0Y}^{(-3)} \quad (3.41)$$

and again

$$k_\phi U_{1N}^{(0)} + k W_{1N}^{(0)} = 0. \quad (3.42)$$

3.4.4. *The fundamental harmonic*: $P_1^{(3)}$, $U_1^{(1)}$, $W_1^{(1)}$, $V_1^{(2)}$

This is the last necessary order of the fundamental harmonic. The jump $C_3^+ - C_3^-$ will be determined by matching $V_1^{(2)}$ to (3.5). From (3.17*a*) we find $P_1^{(3)}$ that is automatically matched to the respective order in (3.4). In the subsequent calculations we will have no need of $P^{(3)}$ (or $P_1^{(4)}$). From (3.17*b*, *c*) at $O(\mu^2)$ and from (3.17*d*) at $O(\mu)$ we find

$$\hat{\mathscr{L}}_1 U_1^{(1)} = -w_c' Y \frac{\partial}{\partial \zeta} \underline{U_1^{(0)}} - \tfrac{1}{2}kw_c'' Y^2 \underline{U_1^{(0)}} - \tfrac{1}{6}\mathrm{i}kw_c''' Y^3 U_1^{(-1)} - \tfrac{1}{2}w_c'' Y^2 \frac{\partial}{\partial \zeta} U_1^{(-1)}$$
$$-\mathrm{i}k_\phi\left(\underline{P_1^{(2)}} + \frac{Y^2}{r_c^2} P_1^{(0)}\right) + \eta\left[\frac{1}{r_c}\underline{U_{1Y}^{(0)}} - \frac{Y}{r_c^2} U_{1Y}^{(-1)} - \left(k^2 + k_\phi^2 + \frac{1}{r_c^2}\right) U_1^{(-1)}\right] + N_U, \quad (3.43)$$

$$\hat{\mathscr{L}}_1 W_1^{(1)} = -w_c' Y \frac{\partial}{\partial \zeta} \underline{W_1^{(0)}} - \tfrac{1}{2}kw_c'' Y^2 \underline{W_1^{(0)}} - \tfrac{1}{6}\mathrm{i}kw_c''' Y^3 W_1^{(-1)} - \tfrac{1}{2}w_c'' Y^2 \frac{\partial}{\partial \zeta} W_1^{(-1)}$$
$$-\mathrm{i}k\underline{P_1^{(2)}} - w_c' V_1^{(2)} - w_c'' Y V_1^{(1)} - \tfrac{1}{2}w_c''' Y^2 V_1^{(0)}$$
$$+\eta\left[\frac{1}{r_c}\underline{W_{1Y}^{(0)}} - \frac{Y}{r_c^2} W_{1Y}^{(-1)} - \left(k^2 + k_\phi^2 + \frac{1}{r_c^2}\right) W_1^{(-1)}\right] + N_W, \quad (3.44)$$

$$V_{1Y}^{(2)} + \frac{1}{r_c} V_1^{(1)} - \frac{1}{r_c^2} Y V_1^{(0)} + \mathrm{i}k_\phi\left(U_1^{(1)} - \frac{Y}{r_c}\underline{U_1^{(0)}} + \frac{Y^2}{r_c^2} U_1^{(-1)}\right) + \mathrm{i}kW_1^{(1)} + \frac{\partial}{\partial \zeta}\underline{W_1^{(0)}} = 0. \quad (3.45)$$

Note that, in addition to nonlinear contributions $N_W$ and $N_U$, the right-hand sides of (3.43) and (3.44) already involve nonlinear contributions in the underlined terms. For $N_U$ and $N_W$ we have

$$N_U = -V_1^{(0)} U_{0Y}^{(-2)} - V_0^{(-1)} U_{1Y}^{(-1)} - \mathrm{i}k_\phi U_0^{(-3)} U_1^{(0)} - \mathrm{i}kW_0^{(-3)} U_1^{(0)} - \frac{1}{r_c} V_1^{(0)} U_0^{(-3)}, \quad (3.46)$$

$$N_W = -V_1^{(0)} W_{0Y}^{(-2)} - V_0^{(-1)} W_{1Y}^{(-1)} - \mathrm{i}k_\phi U_0^{(-3)} W_1^{(0)} - \mathrm{i}kW_0^{(-3)} W_1^{(0)}, \quad (3.47)$$

and
$$k_\phi N_U + kN_W = -V_1^{(0)}(k_\phi U_0^{(-2)} + kW_0^{(-2)}) - \frac{1}{r_c} k_\phi V_1^{(0)} U_0^{(-3)}. \quad (3.48)$$

From (3.43)–(3.45) and (3.48) we obtain the equation for $V_{1YY}^{(2)}$:

$$\hat{\mathscr{L}}_1 V_{1YY}^{(2)} = \mathscr{R}_L + \mathscr{R}_N. \quad (3.49)$$

Here

$$\mathscr{R}_N = \mathrm{i}V_1^{(0)}\left[\left(k_\phi U_0^{(-2)} + kW_0^{(-2)}\right)_{YY} + \frac{1}{r_c} k_\phi U_{0Y}^{(-3)}\right] - (k_\phi^2 + k^2)(P_{1N}^{(2)})_Y$$
$$+\frac{\mathrm{i}}{r_c} k_\phi \frac{\partial}{\partial Y}\left[\hat{\mathscr{L}}_1\left(YU_{1N}^{(0)} - \frac{\partial}{\partial \zeta} W_{1N}^{(0)}\right)\right], \quad (3.50)$$

and we will not give here the unwieldy expression for $\mathscr{R}_L$ because we need only its asymptotic representation in the upper half-plane $Y$ for matching to (3.5). As $Y \to \infty$, from (3.49) we obtain

$$V_{1YY}^{(2)} \sim Q + SY^{-1}, \quad (3.51)$$

where

$$\left.\begin{aligned} Q &= \frac{2\mathrm{i}}{kw_c'}\left\{4(\delta + \alpha\kappa) - \frac{3}{2}\frac{w_c''}{w_c'}\kappa + 2\beta\left[\left(\frac{w_c''}{2w_c'}\right)^2 - \frac{w_c'''}{6w_c'}\right]\right\}, \\ S &= \frac{3\mathrm{i}}{kw_c'} C_4 \end{aligned}\right\} \quad (3.52)$$

(see (3.4) for $C_4$).

By comparing (3.52) with (3.5) and introducing $\tilde{V}_1^{(2)}$ through the relationship $V_1^{(2)} = \tilde{V}_1^{(2)} + \frac{1}{2}QY^2$, for the desired jump $C_3^+ - C_3^-$ we have

$$\frac{3\mathrm{i}}{kw_c'}(C_3^+ - C_3^-) = \int_{-\infty}^{\infty} \tilde{V}_{1YY}^{(2)}\,\mathrm{d}Y. \tag{3.53}$$

We split $\tilde{V}_1^{(2)}$ into its linear and nonlinear parts: $\tilde{V}_1^{(2)} = \tilde{V}_{1L}^{(2)} + \tilde{V}_{1N}^{(2)}$ where

$$\hat{\mathscr{L}}_1(\tilde{V}_{1N}^{(2)})_{YY} = \mathscr{R}_N, \tag{3.54}$$

and $\tilde{V}_{1L}^{(2)} \sim 3\mathrm{i}C_4 Y^{-1}/(kw_c')$ as $|Y| \to \infty$ in the upper half-plane $Y$. Hence the contribution from $\tilde{V}_L$ to the integral (3.53) is $-\mathrm{i}\pi\{3\mathrm{i}C_4/(kw_c')\}$, and we finally obtain

$$C_3^+ - C_3^- = -\mathrm{i}\pi C_4 - \tfrac{1}{3}\mathrm{i}kw_c' \int_{-\infty}^{\infty} (\tilde{V}_{1N}^{(2)})_{YY}\,\mathrm{d}Y, \tag{3.55}$$

and (see (3.11))

$$(C_3^+ - C_3^-)_N = -\tfrac{1}{3}\mathrm{i}kw_c' \int_{-\infty}^{\infty} (\tilde{V}_{1N}^{(2)})_{YY}\,\mathrm{d}Y. \tag{3.56}$$

To complete the derivation of the nonlinear evolution equation (NEE), it only remains for us to calculate a further number of contributions involved in $\mathscr{R}_N$: $W_0^{(-3)}$, $U_0^{(-3)}$, $(k_\phi U_0^{(-2)} + kW_0^{(-2)})$ as well as the contribution from the second harmonic $U_2^{(-3)}$ appearing in $P_{1N}^{(2)}$ (see (3.31), (3.32)).

3.4.5. *The zeroth harmonic:* $U_0^{(-3)}$, $W_0^{(-3)}$, $V_0^{(-1)}$

From (3.17*b*, *c*) at $O(\mu^{-2})$, in view of (3.28) we have

$$\hat{\mathscr{L}}_0 U_0^{(-3)} = -(V_1^{(0)}\overline{U_{1Y}^{(-1)}} + \text{c.c.}); \quad \hat{\mathscr{L}}_0 W_0^{(-3)} = -(V_1^{(0)}\overline{W_{1Y}^{(-1)}} + \text{c.c.}). \tag{3.57}$$

Note that the relationship (3.36) used above follows herefrom. From (3.17*d*)

$$V_{0Y}^{(-1)} = -\frac{\partial}{\partial\zeta}W_0^{(-3)} = \frac{k_\phi}{k}\frac{\partial}{\partial\zeta}U_0^{(-3)}. \tag{3.58}$$

For $U_0^{(-3)}$ and $W_0^{(-3)}$, from (3.57) and (3.58) we get

$$U_{0Y}^{(-3)} = -\frac{2\beta}{kw_c'}k_\phi(kw_c')^2\int_0^\infty \frac{\mathrm{d}x_1}{c}\int_0^\infty \frac{\mathrm{d}x_2}{c}\left(\frac{x_1}{c}\right)^2 \exp\left\{-\frac{\eta}{3}(kw_c')^2\left[\left(\frac{x_1}{c}\right)^3 + 3\left(\frac{x_1}{c}\right)^2\frac{x_2}{c}\right]\right\}$$

$$\times \exp\left[\mathrm{i}\frac{x_1}{c}(\Omega + kw_c' Y)\right]A(\zeta - x_2)\bar{A}(\zeta - x_1 - x_2) + \text{c.c.}, \tag{3.59}$$

$$W_0^{(-3)} = -\frac{k_\phi}{k}U_0^{(-3)}. \tag{3.60}$$

Note that the asymptotic representation of the zeroth harmonic

$$U_0^{(-3)} \sim \frac{2k_\phi}{(kw_c')^3}(k^2 + k_\phi^2)\,Y^{-3}|A|^2 \tag{3.61}$$

is matched to the asymptotic representation of the zeroth harmonic of the outer solution (not written here).

3.4.6. *The zeroth harmonic:* $U_0^{(-2)}, W_0^{(-2)}$

From (3.17*b*, *c*) at $O(\mu^{-1})$ and (3.58) we find

$$\hat{\mathscr{L}}_0(k_\phi U_0^{(-2)}+kW_0^{(-2)})_Y = -\left\{V_1^{(0)}\left[(k_\phi \overline{U_1^{(0)}}+k\overline{W_1^{(0)}})_Y+\frac{k_\phi}{r_c}\overline{U_1^{(-1)}}\right]_Y+\text{c.c.}\right\} -k_\phi w_c' \frac{\partial}{\partial \zeta} U_0^{(-3)}. \quad (3.62)$$

3.4.7. *The second harmonic* $U_2^{(-3)}$

From (3.17*b*) at $O(\mu^{-2})$ we have

$$\hat{\mathscr{L}}_2 U_2^{(-3)} = -V_1^{(0)} U_{1Y}^{(-1)} \quad (3.63)$$

to yield

$$U_2^{(-3)} = 2\mathrm{i}\beta k_\phi \int_0^\infty \frac{dx_1}{c}\int_0^\infty \frac{dx_2}{c}\frac{x_1}{c}\exp\left\{-\frac{\eta}{3}(kw_c')^2\left[4\left(\frac{x_2}{c}\right)^3+\left(\frac{x_1}{c}\right)^3+3\frac{x_1x_2}{c^2}\frac{2x_2+x_1}{c}\right]\right\} \times\exp\left[-\mathrm{i}\frac{x_1+2x_2}{c}(\Omega+kw_c' Y)\right]A(\zeta-x_2)A(\zeta-x_1-x_2). \quad (3.64)$$

### 3.5. *Completion of the derivation of the NEE*

By gathering together all nonlinear contributions, we write the expression (3.50) for $\mathscr{R}_N$ as

$$\mathscr{R}_N = \sum_{n=1}^{10}\mathscr{R}^{(n)}, \quad (3.65)$$

where

$$\mathscr{R}^{(1)} = -\mathrm{i}V_1^{(0)}H^{(1)}, \qquad \hat{\mathscr{L}}_0 H^{(1)} = k_\phi w_c' \frac{\partial}{\partial\zeta}U_{0Y}^{(-3)};$$

$$\mathscr{R}^{(2)} = -\mathrm{i}V_1^{(0)}H^{(2)}, \qquad \hat{\mathscr{L}}_0 H^{(2)} = -\mathrm{i}V_1^{(0)}\overline{V_{1YYYY}^{(1)}};$$

$$\mathscr{R}^{(3)} = -\mathrm{i}V_1^{(0)}H^{(3)}, \qquad \hat{\mathscr{L}}_0 H^{(3)} = 4\frac{k_\phi}{r_c}V_1^{(0)}\overline{U_{1YY}^{(1)}};$$

$$\mathscr{R}^{(4)} = -\mathrm{i}V_1^{(0)}H^{(4)}, \qquad \hat{\mathscr{L}}_0 H^{(4)} = -\mathrm{i}\frac{k_\phi}{r_c}V_1^{(0)}\frac{\partial}{\partial\zeta}\overline{U_{1YYY}^{(-1)}};$$

$$\mathscr{R}^{(5)} = -\mathrm{i}V_1^{(0)}H^{(5)}, \qquad \hat{\mathscr{L}}_0 H^{(5)} = -\frac{2\eta}{r_c}k_\phi U_{0YYY}^{(-3)};$$

$$\mathscr{R}^{(6)} = -\frac{k_\phi}{r_c}\frac{\partial}{\partial\zeta}(V_1^{(0)}U_{0YY}^{(-3)});$$

$$\mathscr{R}^{(7)} = -2\mathrm{i}\frac{k_\phi}{r_c}\eta(U_{1N}^{(0)})_{YY};$$

$$\mathscr{R}^{(8)} = -(k_\phi^2+k^2)(P_{1N}^{(2)})_{Y,0} \equiv -\frac{2}{r_c}(k_\phi^2+k^2)(U_0^{(-3)}U_1^{(-1)})_Y;$$

$$\mathscr{R}^{(9)} = -(k_\phi^2+k^2)(P_{1N}^{(2)})_{Y,2} \equiv -\frac{2}{r_c}(k_\phi^2+k^2)(U_2^{(-3)}\overline{U_1^{(-1)}})_Y.$$

Here $(P_{1N}^{(2)})_{Y,0}$ and $(P_{1N}^{(2)})_{Y,2}$ denote the contributions to $(P_{1N}^{(2)})_Y$ from the zeroth and second harmonics, respectively. All terms that do not make a contribution to the jump, are included in $\mathscr{R}^{(10)}$. In accordance with (3.65), we calculate contributions to the integral (3.56)

$$\int_{-\infty}^{\infty} (\tilde{V}_{1N}^{(2)})_{YY}\, \mathrm{d}Y = \sum_{n=1}^{9} J^{(n)}. \tag{3.66}$$

By designating

$$\tau = \zeta/c, \quad \tau_i = x_i/c, \quad \lambda = \eta(kw_c')^2, \quad k_t^2 = k^2+k_\phi^2, \quad a = \frac{2\pi}{|kw_c'|}(k_t^2 k_\phi)^2,$$

$$A_1 = A(\zeta-x_1), \quad A_2 = A(\zeta-x_1-x_2), \quad A_3 = A(\zeta-2x_1-x_2),$$

we write

$$J^{(1)} = -a\frac{w_c'}{c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^2(1-\lambda\tau_1^2\tau_2)\, E_1(\tau_1,\tau_2)\, A_1 A_2 \bar{A}_3,$$

$$J^{(2)} = -a\frac{2}{3}\frac{\lambda}{r_c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^5 E_1(\tau_1,\tau_2)\, A_1 A_2 \bar{A}_3,$$

$$J^{(3)} = a\frac{4}{r_c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^2 E_1(\tau_1,\tau_2)\, A_1 A_2 \bar{A}_3,$$

$$J^{(4)} = -a\frac{w_c'}{c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^2 E_1(\tau_1,\tau_2)\, A_1 A_2 \left(\tau_1\frac{\partial}{\partial\tau}\right)\bar{A}_3,$$

$$J^{(5)} = -a\frac{2\lambda}{r_c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^4\tau_2 E_1(\tau_1,\tau_2)\, A_1 A_2 \bar{A}_3,$$

$$J^{(6)} = -a\frac{w_c'}{c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_1^2 E_1(\tau_1,\tau_2)\left(\tau_1\frac{\partial}{\partial\tau}\right)(A_1 A_2 \bar{A}_3),$$

$$J^{(7)} = J^{(5)},$$

$$J^{(8)} = a\frac{2}{r_c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, A_1 A_2 \bar{A}_3 \left\{\tau_1 E_1(\tau_1,\tau_1+\tau_2)\int_0^{\tau_1} \mathrm{d}\tau_3\, E_2(\tau_1,\tau_3) + (\tau_1+\tau_2)\, E_1(\tau_1+\tau_2,\tau_1)\int_0^{\tau_1} \mathrm{d}\tau_3\, E_2(\tau_1+\tau_2,\tau_3)\right\},$$

$$J^{(9)} = -a\frac{2}{r_c}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \tau_2 A_1 A_2 \bar{A}_3 E_3(\tau_1,\tau_2)\int_0^{\tau_1} \mathrm{d}\tau_3\, E_4(\tau_1,\tau_2,\tau_3).$$

Here

$$\left.\begin{aligned} E_1(x,y) &= \exp\{-\tfrac{1}{3}\lambda(2x^3+3x^2y)\}, \quad E_2(x,y) = \exp\{-\lambda(xy^2-2x^2y)\},\\ E_3(x,y) &= \exp\{-\tfrac{1}{3}\lambda(x^3+y^3+(x+y)^3)\},\\ E_4(x,y,z) &= \exp\{-\lambda\,[\tfrac{4}{3}z^3+z^2(3y+2x)+z(2y^2+2xy)]\}. \end{aligned}\right\} \tag{3.67}$$

Note that the contributions $J^{(4)}$ and $J^{(6)}$ contain amplitude derivatives with respect to $\tau$. Their origin, as well as that of the contribution $J^{(1)}$, is wholly associated with the spatial formulation of the problem, while they are not present in the temporal problem. These contributions remain in a plane model; the other contributions, however, are due to the finite curvature and disappear as $D \to 0$.

The contributions $J^{(4)}$ and $J^{(6)}$ can be transformed to a form not containing derivatives with respect to $\tau$. To do so, note that

$$\frac{\partial}{\partial\tau}(A_1 A_2 \bar{A}_3)+A_1 A_2 \frac{\partial}{\partial\tau}\bar{A}_3 = -\frac{\partial}{\partial\tau_1}(A_1 A_2 \bar{A}_3).$$

On substituting (3.66) into (3.56) and subsequently into (3.12), we obtain the desired NEE:

$$\frac{J_2}{r_c}\left(\frac{\mathrm{d}A}{\mathrm{d}\zeta}+\mathrm{i}\Omega\frac{J_1}{J_2}A\right) = \int_0^\infty \frac{\mathrm{d}x_1}{c}\int_0^\infty \frac{\mathrm{d}x_2}{c}\mathscr{K}\left(\frac{x_1}{c},\frac{x_2}{c}\right)A(\zeta-x_1)\,A(\zeta-x_1-x_2)\,\bar{A}(\zeta-2x_1-x_2). \tag{3.68}$$

The kernel $\mathscr{K}(\tau_1,\tau_2)$ is

$$\begin{aligned}\mathscr{K}(\tau_1,\tau_2) &= -2\pi(k_t^2 k_\phi/w_c')^2 \\ &\times\left\{\left[\frac{1}{r_c}\left(2\lambda\frac{\partial}{\partial\lambda}+4\right)-\frac{w_c'}{c}\left(3\lambda\frac{\partial}{\partial\lambda}+4\right)\right]\tau_1^2 E_1(\tau_1,\tau_2)\right. \\ &+\frac{2}{r_c}\tau_1 E_1(\tau_1,\tau_1+\tau_2)\int_0^{\tau_1}\mathrm{d}\tau_3\, E_2(\tau_1,\tau_3) \\ &+\frac{2}{r_c}(\tau_1+\tau_2)\,E_1(\tau_1+\tau_2,\tau_1)\int_0^{\tau_1}\mathrm{d}\tau_3\, E_2(\tau_1+\tau_2,\tau_3) \\ &\left.-\frac{2}{r_c}\tau_2 E_3(\tau_1,\tau_2)\int_0^{\tau_1}\mathrm{d}\tau_3\, E_4(\tau_1,\tau_2,\tau_3)\right\}.\end{aligned} \tag{3.69}$$

## 4. Analysis of the evolution equation

### 4.1. *Evolution in the regime of an unsteady CL*

If the initial supercriticality is large enough, such that the spatial growth rate $\gamma_L \gg \nu^{1/3}$, one can expand (3.69) in powers of $\lambda \ll 1$. On passing to 'physical' variables as well as, for the sake of convenience, assuming $x_1+x_2 = \xi$, $x_1 = \xi\sigma$ and introducing the amplitude $B = A\exp(-\mathrm{i}(\Delta k_r z)$, we obtain at the leading order the NEE

$$\frac{\mathrm{d}B}{\mathrm{d}z}-\gamma_L B = \mathrm{e}^{-\mathrm{i}\psi}\mathscr{N}\int_0^\infty \mathrm{d}\xi\,\xi^3\int_0^1 \sigma^2\,\mathrm{d}\sigma\, B(z-\xi)\,B(z-\sigma\xi)\,\bar{B}(z-(1+\sigma)\,\xi), \tag{4.1}$$

where
$$\psi = \arg(J_2/r_c)-\pi,\quad \mathscr{N} = 2\pi|r_c/J_2|\,(2k_t^2 k_\phi/w_c')^2\left(\frac{2}{r_c}+\frac{|w_c'|}{c}\right). \tag{4.2}$$

Other terms in the expansion of (3.69) give only small corrections to (4.1) and can be neglected. Let us emphasize once again the dual origin of the oblique nonlinearity: both the finiteness of the curvature (the term $\sim 2/r_c$ in (4.2)) and the streamwise variation of amplitude (the term $\sim w_c'/c \sim 1/d$) contribute to it, and both of these contributions are of the same sign and are comparable to each other in the case of a not too small curvature.

Equation (4.1) was obtained and investigated in detail by Goldstein & Leib (1989) and Shukhman (1991), who solved quite different problems: the nonlinearity in Goldstein & Leib (1989) was due to a pole-type singularity of a temperature perturbation on the CL, while in Shukhman (1991) it was associated with a logarithmic singularity of the neutral mode on the CL which arises in a two-dimensional problem

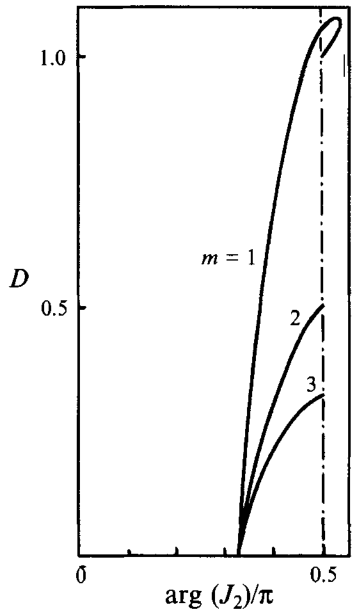


FIGURE 3. The dependence of $\arg(J_2)$ on $D$ for $m = 1, 2, 3$.

if the radiation condition at infinity (in $y$ or in $r$) is imposed rather than one of vanishing of perturbation (or if there are several CLs simultaneously). The character of the disturbance evolution governed by (4.1) is defined by the only parameter, the phase $\psi$. A growth of amplitude $B$, after the nonlinearity threshold is reached (when the nonlinear term becomes of order $\gamma_L B$, i.e. competitive)

$$B \sim B_N = O(\gamma_L^{5/2}) \tag{4.3}$$

proceeds explosively:

$$B \sim (z_0 - z)^{-5/2+i\beta(\psi)}. \tag{4.4}$$

The dependence $\beta(\psi)$ was investigated by Goldstein & Leib (1989) and Shukhman (1991). When $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$ this dependence is single-valued, and the asymptotic representation of the solution of NEE (4.1) is indeed described by (4.4). When $\frac{1}{2}\pi < |\psi| < \pi$ the function $\beta(\psi)$ becomes two-valued. In view of the nonlinear character of (4.1) the asymptotic behaviour of $B(z)$ (as $z \rightarrow z_0$) is now described neither by (4.4), nor by a sum of two terms of the form (4.4), but – as shown by the numerical calculations of these authors – remains explosive as before.

In the model (2.1) as $D \rightarrow 0$:

$$J_1 = 2i\pi, \quad J_2/r_c = 2(2+i\pi), \quad \arg(J_2/r_c) = \arctan(\tfrac{1}{2}\pi) \approx 0.32\pi.$$

The dependence of $\arg(J_2/r_c)$ on $D$ is calculated numerically and is presented in figure 3. One can see that (except for a small region for the mode $m = 1$) $\arg(J_2/r_c)$ varies in the range from $0.32\pi$ as $D \rightarrow 0$ to $\frac{1}{2}\pi$ as $D \rightarrow 1/m$, hence $\psi$ lies in the region $-\pi < \psi < -\frac{1}{2}\pi$.

### 4.2. *The evolution in the regime of a viscous CL*

#### 4.2.1. *NEE: local and non-local nonlinearity*

Let us obtain the NEE in the regime of a viscous CL, $\gamma \ll \nu^{1/3}$, where $\gamma \sim A^{-1}\,\mathrm{d}A/\mathrm{d}z$ is a local spatial growth rate. Formally, this is achieved through an expansion in terms of the small parameter $\lambda^{-1}$. At first glance, all 'delays' should be put equal to zero (because there is no 'memory' in the case of a 'large' viscosity), and all amplitudes should be carried outside the integral sign. It is easy to perform the remaining integrations, and we obtain a nonlinear term of the form $\sim A|A|^2\lambda^{-4/3}$. This is actually the main term of the expansion, but it can be small and go to zero at some values of the parameters, so the next term then becomes important. Such a situation occurs in our problem in the case of a small curvature. It appears that the next term of the expansion is of $O(A^3\lambda^{-4/3}(\gamma/\lambda^{1/3})^{1/2})$, i.e. it contains – with respect to the main term – a factor $(\gamma/\nu^{1/3})^{1/2} = (l_t/l_\nu)^{1/2} \ll 1$ (see (1.2)). Contributions at this order are made only by the first two of the four terms involved in the right-hand side of (3.69). The exponents in them that depend on $\tau_1$ and $\tau_2$ are linear in $\tau_2$ and therefore the $\tau_2$-associated delay must be taken into account. The result is

$$(\mathrm{d}/\mathrm{d}z-2\gamma_L)|A(z)|^2 = 4\pi\,\mathrm{Re}\,(r_c/J_2)(k_t^2k_\phi/w_c')^2$$
$$\times\left\{-\tilde{\lambda}^{-4/3}\frac{I}{r_c}|A(z)|^4+\tilde{\lambda}^{-3/2}\left(\frac{3}{r_c}+\frac{w_c'}{2c}\right)\frac{(\pi c)^{1/2}}{2}|A(z)|^2\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty \mathrm{d}\zeta\zeta^{-1/2}|A(z-\zeta)|^2\right\}, \quad (4.5)$$

where $\tilde{\lambda} = \nu(kw_c')^2$, and

$$I = (2/3)^{2/3}\Gamma(\tfrac{1}{3})\left[\tfrac{2}{3}+\frac{2}{5^{1/3}}F(\tfrac{1}{3},\tfrac{1}{2},\tfrac{3}{2};-\tfrac{3}{5})-2^{1/3}I_0\right], \quad (4.6)$$

$$I_0 = 3^{-4/3}\int_0^1\sigma(1-\sigma)\,\mathrm{d}\sigma\int_0^1\mathrm{d}\rho\{\tfrac{2}{3}-\sigma(1-\sigma)+\sigma\rho[\tfrac{4}{3}\sigma^2\rho^2+\sigma\rho(3-\sigma)+2(1-\sigma)]\}^{-4/3}.$$

Here $F$ and $\Gamma$ are, respectively, a hypergeometric function and a gamma-function. Calculations yield $I_0 = 0.05443$ and $I = 3.48350$.

Thus, the nonlinear term in the regime of a viscous CL contains two contributions, local and non-local. The local contribution is entirely due to the curvature of the shear layer and vanishes as the curvature tends to zero. In this case the non-local term in (4.5) becomes a leading one.

If the curvature is not small, i.e. $D = O(1)$, the non-local nonlinearity is not competitive compared with the local one, and we obtain the usual Landau–Stuart–Watson equation with a stabilizing nonlinearity because $\mathrm{Re}\,(r_c/J_2) > 0$ (see figure 3). The amplitude reaches saturation at the level

$$A_{sat} \approx (\gamma_L\,\nu^{4/3})^{1/2}. \quad (4.7)$$

#### 4.2.2. *The limit of a plane model*

In the limiting case of a plane model the NEE has the form

$$\frac{\mathrm{d}|A(z)|^2}{\mathrm{d}z} = 2\gamma_L|A(z)|^2-b_1\tilde{\lambda}^{-3/2}|A(z)|^2\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty \mathrm{d}\zeta\zeta^{-1/2}|A(z-\zeta)|^2, \quad (4.8)$$

where
$$b_1 = -\pi(\pi c)^{1/2}(w_c'/c)\,\mathrm{Re}\,(r_c/J_2)(k_t^2k_\phi/w_c')^2. \quad (4.9)$$

Recently an analogous equation

$$\frac{\mathrm{d}|A(t)|^2}{\mathrm{d}t} = 2\gamma_L|A(t)|^2+g|A(t)|^2\int_0^\infty \mathrm{d}\tau|A(t-\tau)|^2, \quad (4.10)$$

was obtained (Smith & Blennerhassett 1992; Wu *et al.* 1993) for the temporal evolution of a disturbance in the form of a pair of oblique waves but with a somewhat different structure of the non-local nonlinear term (cf. (4.8)). The origin of a non-local nonlinearity is associated with the dynamics of the intermediate region (connecting the CL region with an external flow) where viscous and unsteady terms in the equations become of the same order of magnitude.† The nonlinearity in (4.10), as demonstrated by Wu *et al.* (1993), depending on the sign of $g$, leads to either an explosive growth of a disturbance or its exponential decay.

A different picture is provided by NEE (4.8): irrespective of the sign of $b_1$, it describes the growth of a disturbance (in our problem $b_1 > 0$, but we shall consider both signs). As $b_1 > 0$, after the disturbance reaches the nonlinearity threshold,

$$A = A_N \sim (\gamma_L^{1/2} \nu^{3/2})^{1/2}, \tag{4.11}$$

the exponential growth is replaced with a slower power-law one

$$|A|^2 \approx \frac{4}{\pi b_1} \gamma_L \tilde{\lambda}^{3/2} z^{1/2} \sim O(\gamma_L \nu^{3/2} z^{1/2}), \tag{4.12}$$

which proceeds in the 'quasi-stationary' regime (when the left-hand side of (4.8) is much less than each of the terms in the right-hand side). If there is no stabilizing factor whatsoever, the disturbance, by reaching amplitude $A \sim \nu^{2/3}$, moves into the regime of a nonlinear CL.

As $b_1 < 0$, the exponential growth, after reaching the threshold of nonlinearity (4.11), is replaced with an explosive one. This a strongly unsteady process, to which there corresponds a balance of the evolution (d/d$z$) and nonlinear terms in (4.8). We find

$$|A|^2 = \frac{\Gamma^2(\frac{1}{4})}{(2\pi)^{1/2}} \frac{1}{b_1} \tilde{\lambda}^{3/2} (z_0 - z)^{-1/2}. \tag{4.13}$$

The growth rate increases: $\gamma \sim O(A^4/\nu^3)$, and when $\gamma \sim \nu^{1/3}$, i.e. as the amplitude reaches $A = O(\nu^{5/6})$, there is a transition to the regime of an unsteady CL. We shall not investigate the case $b_1 < 0$ in more detail because in (4.8) $b_1 > 0$ (although the multiplier $(3/r_c + w_c'/2c)$, involved in (4.5) does change its sign in the case of a rather small curvature ($D \approx 1/7$ in the model (2.1)), the local stabilizing nonlinearity at such $D$ is already essential, and the explosive regime (4.13), described above, is not realized).

### 4.3. *The picture of the evolution as a whole*

Let us represent the evolution scenarios obtained on an amplitude–supercriticality diagram. It is convenient to write the NEE (3.68) in a symbolic form

$$\frac{dA}{dz} - \gamma_L A = \left( -c_1 D - c_2 \left( \frac{l_t}{l} \right)^{1/2} \right) \frac{A^3}{l^4}; \tag{4.14}$$

$c_1$ and $c_2$ are coefficients of order unity, and $l$ is the CL scale (either unsteady $l_t$ or viscous $l_\nu$, see (1.2)). This equation reproduces all the relevant properties of an exact NEE (3.68) and would be quite sufficient for a qualitative analysis.

#### 4.3.1. *The evolution at $D = O(1)$*

A diagram for the case $D = O(1)$ is given in figure 4. If supercriticality is small, $\gamma_L \ll \nu^{1/3}$, the disturbance starts from the region of a viscous CL and does not leave it throughout the evolution up to saturation at $A_{sat} \sim \gamma_L^{1/2} \nu^{2/3}$. At larger supercriticality,

† Wu *et al.* (1993) have called this region the 'diffusion layer'. But in critical-layer literature there is already an analogous term: the layers near the 'cat's eyes' boundary are called 'diffusive layers'. In view of this we prefer to use the term 'intermediate region'.

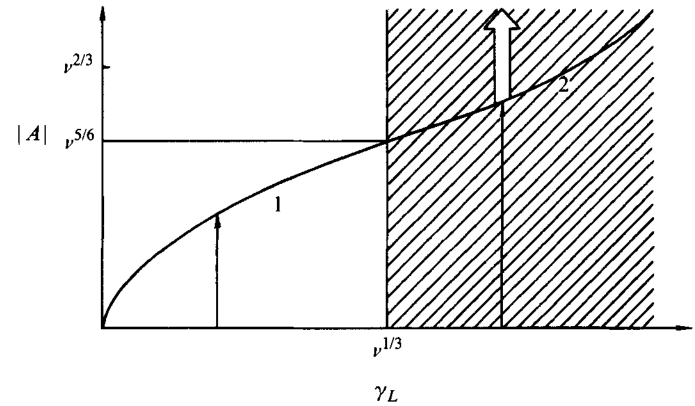


FIGURE 4. The amplitude–supercriticality diagram for the case $D = O(1)$. The region of an unsteady CL is shaded. Curve 1, threshold of nonlinearity for a viscous CL, it also represents the level of saturation $A = A_1 \sim (\gamma_L \nu^{4/3})^{1/2}$; curve 2, threshold of nonlinearity for an unsteady CL: $A = A_2 \sim \gamma_L^{5/2}$. The vertical arrows indicate the various evolutionary stages:

$$\rightarrow - A \sim \exp(\gamma_L z), \Rightarrow - A \sim (z_0 - z)^{-5/2}.$$

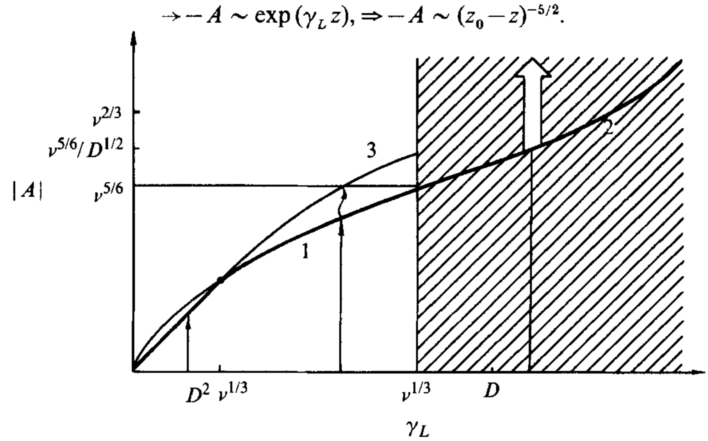


FIGURE 5. The amplitude–supercriticality diagram for the case $\nu^{1/3} \ll D \ll 1$. The threshold of nonlinearity is shown by a heavy line. Curve 3 corresponds to the level of saturation in the regime of a viscous CL. The equations for the curves are: 1, $A = A_1 \sim (\gamma_L \nu^3)^{1/4}$; 2, $A = A_2 \sim \gamma_L^{5/2}$; 3, $A = A_3 \sim (\gamma_L \nu^{4/3}/D)^{1/2}$. The wavy arrow denotes a power-law growth: $|A| \sim z^{1/4}$.

$\gamma_L > \nu^{1/3}$, the disturbance starts from the region of an unsteady CL and, by reaching the threshold of nonlinearity (4.3), switches to the explosive regime $|A| \sim (z_0 - z)^{-5/2}$. The growth rate grows fast, and the unsteady scale $l_t \sim \gamma = A^{2/5}$ always remains larger than the nonlinear one, $l_N \sim A^{1/2}$, such that the nonlinear CL regime is not realized.

### 4.3.2. *The evolution at $D \ll 1$*

In the case $D \ll 1$ the picture in the region of an unsteady CL remains unchanged. However, in the region of small supercriticalities, $\gamma_L \ll \nu^{1/3}$, the evolution appears different. If $D > \nu^{1/3}$, the local nonlinearity is capable of stabilizing the growth of the disturbance in the regime of a viscous CL throughout the region $0 < \gamma_L < \nu^{1/3}$ (see figure 5). If, however, $D < \nu^{1/3}$, in the range of supercriticalities $D < \gamma_L < \nu^{1/3}$ this stabilizing effect turns out to be insufficient, and the evolution continues (at $A > \nu^{2/3}$) in the regime of a nonlinear CL. The transition to the regime of a nonlinear CL and the subsequent evolution calls, generally speaking, for a separate analysis, but we will try to predict the result qualitatively. It seems likely that after transition the amplitude would also grow with a power-like law but with a different power than in the regime of a viscous CL (see (4.12)). We shall obtain the desired power from the following considerations. In the regime of a nonlinear CL, rearrangement of the flow inside the CL provides a reduction of the linear growth rate (see, for example, Huerre & Scott 1980; Churilov & Shukhman 1987): $\gamma_L \to (\gamma_L \nu / A^{3/2})$. On the other hand, the non-local nonlinear term (see (4.8)) that has the order $(\gamma/\nu^{1/3})^{1/2}\, \nu^{-4/3} A^3$ originates not from the CL but from the intermediate region, and it seems to be unchanged. Therefore, a model NEE has the form

$$\frac{\mathrm{d}A}{\mathrm{d}z} = \left(\gamma_L \frac{\nu}{A^{3/2}} - c_2 \frac{\gamma^{1/2}}{\nu^{3/2}} A^2\right) A. \tag{4.15}$$

From the balance of the terms involved in the right-hand side we find $\gamma \sim \nu^5 \gamma_L^2 A^{-7}$, which yields

$$A \sim (\gamma_L^2 \nu^5)^{1/7} z^{1/7}. \tag{4.16}$$

This is a very slow growth, and it will necessarily stop by reaching a certain amplitude if some stabilizing or dissipative factor is further taken into account. In §5 the influence on the perturbation development of such a factor, a viscous broadening of the jet, is discussed in detail.

## 5. Discussion

Thus, we have demonstrated that for $\gamma_L > \nu^{1/3}$ the oblique nonlinearity, even in the case of a single wave, leads to an explosive growth, $|A| \sim (z_0 - z)^{-5/2}$, due to both the finite curvature and the streamwise variation of amplitude. This growth continues up to amplitudes $A = O(1)$, where the weakly nonlinear theory becomes invalid.

In our reasoning it was implicitly assumed that the coefficient in the nonlinear term (3.68), proportional to $k_\phi^2 ((k_\phi d)^2$ in a dimensional form), is of order unity. However, if curvature $D$ is small and azimuthal number $m$ is not large, the oblique nonlinearity is reduced by factor $\sim (mD)^2$. Indeed, $(k_\phi d)^2 = (m/r_c)^2 \sim (md/R)^2 = (mD)^2$. Therefore, when $(mD)^2 \ll 1$ (and for a plane problem when $(k_\perp d)^2 \ll 1$†) taking into account the two-dimensional nonlinearity becomes important. Let us analyse to what this would lead. We begin with the region $\gamma_L \gg \nu^{1/3}$. In the regime of an unsteady CL the evolution equation can be written in a symbolic form

$$\frac{\mathrm{d}A}{\mathrm{d}z} = \gamma_L \left(1 - c_3 \frac{A^2}{\gamma_L^4}\right) A + c_2 (mD)^2 \frac{A^3}{\gamma^4}. \tag{5.1}$$

Here the term containing $c_3$ describes a two-dimensional nonlinearity (nonlinear reducing of the growth rate, see, for example, Churilov & Shukhman 1987; see also (5.4) and comments on it).

From (5.1) it is evident that oblique nonlinearity is important only if $\gamma_L \ll (mD)^2$. In the case of a larger supercriticality, it is unimportant, and the evolution of quasi-two-dimensional disturbances does not differ from the evolution of two-dimensional disturbances. Figure 6 shows part of the amplitude–supercriticality diagram for quasi-

† Note that in a plane problem the temporal evolution in the case of small $k_\perp d$ was treated by Wu (1993) who considered a disturbance modulated in the spanwise direction, i.e. having a continuum in the $k_\perp$ spectrum. In the case of an axial jet the discreteness of $m$ does not admit such an approach. Note that the NEE obtained by Wu (1993) in the regime of a viscous CL also has a non-local nonlinearity but of a rather different structure.

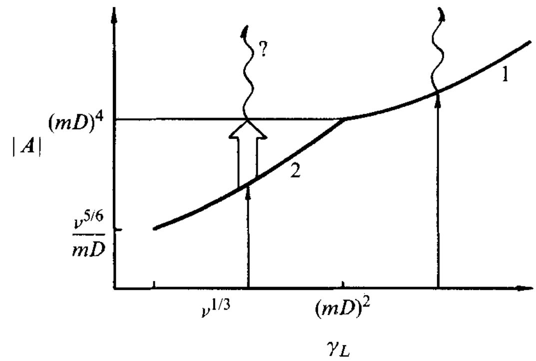


FIGURE 6. The amplitude–supercriticality diagram for quasi-two-dimensional disturbances $(mD)^2 \ll 1$. Curve 1 ($A = A_1 \sim \gamma_L^2$) and curve 2 ($A = A_2 \sim \gamma_L^{5/2}/(mD)$) are thresholds of nonlinearity. The wavy arrow denotes a power-law growth, $A \sim z^{2/3}$, while the wavy arrow labelled ? denotes a power-law growth with an exponent that is not yet established.

two-dimensional disturbances for not too small $(mD)^2$: $\nu^{1/3} < (mD)^2 \ll 1$ in the region $\gamma_L > \nu^{1/3}$. If $\gamma_L > (mD)^2$, the disturbance grows exponentially to the nonlinearity threshold $A \sim A_1 \approx \gamma_L^2$ and then moves into the regime of a nonlinear CL while continuing to grow as a power law, $A \sim z^{2/3}$. If, however, $\gamma_L < (mD)^2$, the disturbance, on reaching the level $A \sim A_2 \approx \gamma_L^{5/2}/(mD)$, begins to grow explosively, $A \sim (z_0 - z)^{-5/2}$. However, although the growth rate increases

$$\gamma \sim [(mD)^2 A^2]^{1/5}, \tag{5.2}$$

this increase is insufficiently fast for the regime of an unsteady CL to become self-maintaining, i.e. in terms of weakly nonlinear theory ($A \ll 1$) no transition to the regime of a nonlinear CL would occur. Indeed, from (5.2) it follows that when

$$A \sim A^* = (mD)^4 \tag{5.3}$$

the nonlinear scale $l_N = A^{1/2}$ would exceed the unsteady scale $l_t = \gamma$, and the disturbance would switch to the regime of a nonlinear CL. The transition to the regime of a nonlinear CL from the explosive growth stage has not been studied previously and calls for a separate consideration. In figure 6 the arrow corresponding to this stage is labelled with a question-mark.

In the region $\gamma_L < \nu^{1/3}$, in addition to the two-dimensional nonlinearity, a further effect, a viscous broadening of the flow, influences the evolution of quasi-two-dimensional ($mD \ll 1$) disturbances. Goldstein & Hultgren (1988, hereinafter referred to as GH), by considering an example of two-dimensional disturbances in a plane shear flow, showed that such a broadening leads to a stabilization of the flow with respect to a given disturbance at some distance $z = z_n$ from the place of jet outflow. (Note that $z_n$ depends on disturbance characteristics, in particular, on supercriticality.) As a result of this stabilization, the disturbance reaches, when $z = z_n$, a maximum amplitude and then decays until it completely disappears.

In the Appendix, for the case $m = 0$ (a good approximation for $mD \ll 1$), a NEE (A 8) is derived which takes into account both a viscous broadening and curvature, and permits us to consider the case $mD \ll 1$, $\gamma_L < \nu^{1/3}$ in a self-consistent manner. Moreover, equation (A 8) is valid not only in the regime of a nonlinear CL (like a corresponding NEE in GH) but also in the regime of a viscous CL, as well as at the stage of transition between them. Hence, it permits us to consider the whole evolution of the perturbation, beginning from its early (linear) stage, and to generalize the results reported by GH.

For studying the evolution of quasi-two-dimensional disturbances on a slowly broadening jet it is convenient to use a NEE (A 8) in the form (A 9):

$$\frac{dA}{dz} = \Phi\left(\frac{\nu}{A^{3/2}}\right)(\gamma_L - \sigma\nu z)A. \tag{5.4}$$

This equation is the generalized (and corrected, see GH) NEE derived by Huerre & Scott (1980). We see in it two physical effects which determine the perturbation evolution. First, the linear growth rate decreases linearly in $z$ due to a broadening. Secondly, the nonlinear reduction of this growth rate (by factor $\Phi$) takes place with an increase of amplitude, as usual. The function $\Phi(s)$ can be calculated analytically for $s \ll 1$ and $s \gg 1$ (see Appendix):

$$\Phi(s) = \begin{cases} bs, & s \ll 1 \\ 1, & s \gg 1, \end{cases}$$

and numerically for arbitrary $s > 0$, and it is reasonably well approximated by the simple formula

$$\Phi(s) = \frac{bs}{1+bs}.$$

On substituting it into (5.4), we can get an analytic solution

$$\ln\frac{A}{A_0} + \frac{2}{3b\nu}(A^{3/2} - A_0^{3/2}) = \gamma_L z - \tfrac{1}{2}\sigma\nu z^2, \tag{5.5}$$

where $A_0 = A(0)$ is the initial amplitude. From (5.5) it is evident that the amplitude reaches a maximum $A = A_{max}$ when $z = z_n = \gamma_L/(\sigma\nu)$,

$$\ln\frac{A_{max}}{A_0} + \frac{2}{3b\nu}(A_{max}^{3/2} - A_0^{3/2}) = \frac{\gamma_L^2}{\sigma\nu}, \tag{5.6}$$

and then decreases to zero. It is easy to see that if $z_n$ is less than the inverse growth rate, $\gamma_L^{-1}$, the dissipation starts almost immediately, following an insignificant growth in amplitude. Therefore, when $\gamma_L < \nu^{1/2}$, there is no cause for speaking of an instability, so fast is the flow stabilized by a viscous broadening. It is interesting to note that an amplification of the disturbance, $A_{max}/A_0$ will also be insignificant when $\gamma_L > \nu^{1/2}$ if its initial amplitude is sufficiently large: $A_0 \gg \gamma_L^{4/3}$. In the intermediate case ($\gamma_L > \nu^{1/2}$, $A_0 \ll \gamma_L^{4/3}$) the amplification is essential.

Figure 7 demonstrates the dependence of $A_{max}$ on $A_0$ for $\gamma_L > \nu^{1/2}$. As long as the initial amplitude is small enough, the coefficient of amplification is constant and is $K_0 = \exp\{\gamma_L^2/(\sigma\nu)\}$, and the entire disturbance evolution proceeds in the regime of a viscous CL. When $A_0 K_0 > \nu^{2/3}$, part of the disturbance evolution now proceeds in the regime of a nonlinear CL, and the coefficient of amplification decreases with an increase of this part. As a result, in the region $\gamma_L^{4/3}/K_0 < A_0 < \gamma_L^{4/3}$, $A_{max}$ almost does not depend on $A_0$: $A_{max} \sim \gamma_L^{4/3}$ (it is this region to which the results of GH pertain). Finally, when $A_0 > \gamma_L^{4/3}$ the amplification coefficient, as has already been pointed out, is virtually unity.

We wish to note in conclusion that a viscous broadening of the flow, when $mD \ll 1$, is in a sense a fateful factor that does not permit the disturbance to reach an amplitude of order unity. Indeed, it has been shown above that when $(mD)^2 \ll 1$, even without taking the viscous broadening into account, the disturbance either is stabilized at a low level or passes to the regime of a nonlinear CL where its growth is heavily

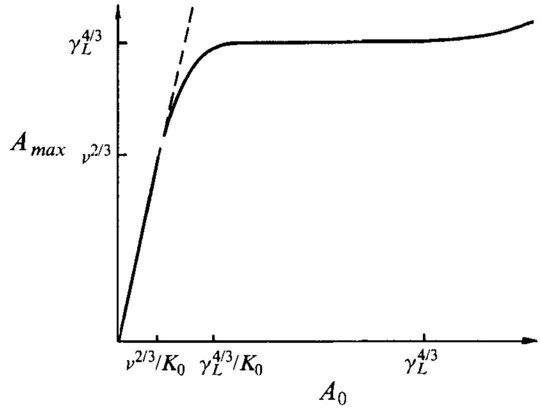


FIGURE 7. The dependence of the maximum achievable amplitude $A_{max}$ on the initial amplitude $A_0$ for a disturbance that starts from the region of small supercriticality, $\nu^{1/2} < \gamma_L < \nu^{1/3}$, when the viscous broadening of the unperturbed flow is taken into account.

retarded (becomes power-law). Owing to this, the broadening effect at any $\gamma_L$ has time to stop the perturbation growth at small amplitudes. The only case when the amplitude reaches $O(1)$ is an explosive development when $mD = O(1)$ and $\gamma_L > \nu^{1/3}$.

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## Appendix. The nonlinear evolution equation for $m = 0$

In view of the two-dimensional character of the problem, it is convenient to introduce the stream function $\psi$: $v_r = -\partial\psi/\partial z$, $v_z = \hat{\mathscr{D}}\psi$ and to deal with one equation:

$$\frac{\partial}{\partial t}\tilde{\Delta}\psi + w\frac{\partial}{\partial z}\tilde{\Delta}\psi - \left(w'' - \frac{w'}{r}\right)\frac{\partial\psi}{\partial z} + \frac{\partial}{\partial z}(\tilde{\Delta}\psi\hat{\mathscr{D}}\psi) - \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial z}\tilde{\Delta}\psi\right) = \nu\tilde{\Delta}^2\psi, \qquad \text{(A 1)}$$

where
$$\tilde{\Delta}F = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}rF\right) + \frac{\partial^2 F}{\partial z^2} + 2\mu\frac{\partial^2 F}{\partial z\,\partial\zeta} + \mu^2\frac{\partial^2 F}{\partial\zeta^2}; \quad \hat{\mathscr{D}}F = \frac{1}{r}\frac{\partial}{\partial r}rF.$$

Moreover, the solution of the problem is conveniently constructed in a real form. Thus, at $O(\epsilon)$ of the outer problem

$$\psi_1^{(1)} = 2B(\zeta)\,\psi_a(r)\cos\theta, \quad \theta = kz - \omega t - \mu\omega_1 t + \Theta(\zeta).$$

Here $\psi_a$ is essentially an eigenfunction of equation (2.4) (for $m = 0$) so that

$$B = \frac{A}{w_c'}; \quad \psi_a = \frac{w_c'\phi_a'}{k^2(c-w)}.$$

We wish to take two factors into account simultaneously: the transition to the regime of a nonlinear CL and a viscous broadening of the flow. The transition is realized when $l_\nu \sim l_N$, which imposes the condition $\nu = O(\epsilon^{3/2})$. A viscous broadening of the external flow provides a contribution to the zeroth harmonic:

$$\delta\psi_0 = \nu\mu^{-1}\zeta u(r) + \nu^2\mu^{-2}\zeta^2 u_1(r) + \nu^3\mu^{-3}\zeta^3 u_2(r) + \dots,$$

where
$$u(r) = \frac{w(r)}{r}\int_0^r \frac{\mathrm{d}s(sw)'}{w^2} \tag{A 2}$$
(the explicit form of the other $u_n$ is not needed for subsequent calculations). A variation of the growth rate caused by the broadening, is $\delta\gamma_L \sim \delta\psi_0 = O(\nu\mu^{-1}\zeta)$. The character of the evolution will change substantially if $\delta\gamma_L \sim \gamma_L = O(\mu)$, i.e. $\mu = O(\nu^{1/2})$. We get the scaling
$$\nu = \eta\epsilon^{3/2}, \quad \mu = \epsilon^{3/4} = O(\nu^{1/2}). \tag{A 3}$$
Otherwise the procedure of constructing the outer solution is standard, and here we give only its inner asymptotic representation (cf. (3.5)):
$$\begin{aligned}\psi+\psi_{00} = {} & \eta\epsilon^{3/4}\zeta u_c + \epsilon(\tfrac{1}{2}w_c' Y^2 + 2B\cos\theta) + \eta\epsilon^{5/4}\zeta u_c' Y + \epsilon^{3/2}(2\alpha_1 BY\cos\theta \\ & + \eta^2\zeta^2 u_{1c}) + \epsilon^{7/4}\eta[(a_\nu^\pm\cos\theta + b_\nu^\pm\sin\theta)\zeta + \tfrac{1}{2}\zeta u_c'' Y^2] \\ & + \epsilon^2[(w_c''' + 3w_c'/r_c^2)Y^4/24 + 2\alpha_2 Y^2 B\cos\theta + a_2^{(2)\pm}\cos 2\theta + \eta^2\zeta^2 u_{1c}' Y] \\ & + \epsilon^{9/4}\Big\langle \eta\zeta(a_\nu^\pm\cos\theta + b_\nu^\pm\sin\theta)\alpha_1 Y + \tfrac{1}{6}\eta\zeta u_c''' Y^3 + \eta^3\zeta^3 u_{2c} \\ & + 2\left\{\frac{w_c'''}{w_c'^2}\left[\left(c_1 - \frac{c}{k}\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}\right)B\cos\theta - \frac{c}{k}\frac{\mathrm{d}B}{\mathrm{d}\zeta}\sin\theta\right]\right. \\ & \left. + \left[\eta\frac{r_c}{cw_c'}\zeta B\cos\theta w_c^{\mathrm{iv}} - w_c'''\left(\frac{w_c'}{c} + \frac{2}{r_c}\right)\right]\right\} Y\ln(\epsilon^{1/2}|Y|) + b^\pm Y\Big\rangle + \dots,\end{aligned}$$
where
$$r - r_c = \epsilon^{1/2}Y, \quad \psi_{00} = \frac{1}{r}\int_{r_c}^r \mathrm{d}s\, s(w-c), \quad c_1 = -\frac{\Omega}{k}, \quad f_c \equiv f(r_c),$$
$$\alpha_1 = -\left(\frac{1}{2r_c} + \frac{3\kappa}{k^2}\right), \quad \alpha_2 = \frac{1}{2}\frac{w_c'''}{w_c'} + \frac{3}{2}\frac{\kappa}{r_c k^2} + \frac{3}{4r_c^2} + \frac{k^2}{2},$$
as well as a new MSC:
$$\begin{aligned}b^+ - b^- = {} & -2\left[\left(c_1 - \frac{c}{k}\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}\right)B\cos\theta - \frac{c}{k}\frac{\mathrm{d}B}{\mathrm{d}\zeta}\sin\theta\right] I_1 \\ & - 4k\left(\frac{\mathrm{d}B}{\mathrm{d}\zeta}\sin\theta + \frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}B\cos\theta\right) I_2 - 2\eta\zeta I_3 B\cos\theta,\end{aligned} \tag{A 4}$$
where
$$I_1 = \frac{1}{r_c}\int_0^\infty \frac{(w'' - w'/r)}{(w-c)^2}\psi_a^2 r\,\mathrm{d}r, \quad I_2 = \frac{1}{r_c}\int_0^\infty \psi_a^2 r\,\mathrm{d}r,$$
$$I_3 = \frac{1}{r_c}\int_0^\infty \frac{\psi_a^2 r\,\mathrm{d}r}{(w-c)^2}\left[(\hat{\mathscr{D}}u)'' - \frac{1}{r}(\hat{\mathscr{D}}u)' - \frac{w'' - w'/r}{w-c}(u' + u/r)\right].$$

The inner solution is also constructed in the usual fashion. Namely, the function $\Psi = (\psi + \psi_0)\epsilon^{-1}$ is represented as an expansion
$$\Psi = \epsilon^{-1/4}\Psi^{(0)} + \Psi^{(1)} + \epsilon^{1/4}\Psi^{(2)} + \epsilon^{1/2}\Psi^{(3)} + \epsilon^{3/4}\Psi^{(4)} + \epsilon\Psi^{(5)} + \epsilon^{5/4}\Psi^{(6)} + \dots. \tag{A 5}$$
The MSC (A 4) is reduced to a system of two evolution equations
$$\left(\frac{c}{k}I_1 - 2kI_2\right)\frac{\mathrm{d}B}{\mathrm{d}\zeta} = \int \mathrm{d}Y\langle S\sin\theta\rangle, \tag{A 6a}$$
$$\left(\frac{c}{k}I_1 - 2kI_2\right)B\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta} - (c_1 I_1 + \eta\zeta I_3)B = \int \mathrm{d}Y\langle S\cos\theta\rangle, \tag{A 6b}$$

where
$$\oint \mathrm{d}Y\langle\ldots\rangle = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}Y\int_{-\pi}^{\pi}\mathrm{d}\theta(\ldots),\quad S=\Psi_{YY}.$$

The main contribution to the right-hand sides of (A 6*a*) and (A 6*b*) is made by the term $O(\epsilon^{5/4})$ of the expansion (A 5). For $\tilde{S}=\Psi^{(6)}_{YY}-\eta\zeta w_c''' Y$ we have the equation

$$w_c' Y\frac{\partial\tilde{S}}{\partial z}+2kB\sin\theta\frac{\partial\tilde{S}}{\partial Y}$$
$$=\eta\tilde{S}_{YY}-\frac{2w_c'''}{w_c'}\left[c\frac{\mathrm{d}B}{\mathrm{d}\zeta}\cos\theta+\left(c_1k-c\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}\right)B\sin\theta\right]-2\rho w_c'\eta\zeta kB\sin\theta,$$

where
$$\rho=\frac{1}{c}\left(\frac{w_c^{\mathrm{iv}}}{w_c'}-\frac{2w_c'''}{w_c' r_c}-\frac{w_c'''}{c}\right),$$

from which we find the quantities involved in the right-hand sides of (A 6*a, b*):

$$\oint \mathrm{d}Y\langle\tilde{S}\sin\theta\rangle=-\left\{\frac{w_c'''}{w_c'^2}\left(c_1-\frac{c\,\mathrm{d}\theta}{k\,\mathrm{d}\zeta}\right)+\eta\zeta\rho\right\}\Phi_1(\lambda)\,B,$$
$$\oint \mathrm{d}Y\langle\tilde{S}\cos\theta\rangle=-\frac{cw_c'''}{kw_c'^2}\Phi_2(\lambda)\frac{\mathrm{d}B}{\mathrm{d}\zeta};\quad \lambda\equiv\frac{\eta}{(2B)^{3/2}}\frac{|w_c'|^{1/2}}{k},\tag{A 7}$$

where the functions

$$\Phi_1(\lambda)=\oint\mathrm{d}y\,\langle g_1\sin\theta\rangle,\quad \Phi_2(\lambda)=\oint\mathrm{d}y\,\langle g_2\cos\theta\rangle$$

and $g_1$ and $g_2$ satisfy the equation

$$\left(\lambda\frac{\partial^2}{\partial y^2}+y\frac{\partial}{\partial\theta}-\sin\theta\frac{\partial}{\partial y}\right)\begin{pmatrix}g_1\\g_2\end{pmatrix}=2\begin{pmatrix}\sin\theta\\\cos\theta\end{pmatrix}.$$

Of practical interest are the asymptotic expansions of $\Phi_1$ and $\Phi_2$:

$$\Phi_1(\lambda)=\begin{cases}C\lambda+O(\lambda^2), & \lambda\ll 1\\ -\pi+a_1\lambda^{-4/3}+O(\lambda^{-8/3}), & \lambda\gg 1,\end{cases}$$
$$\Phi_2(\lambda)=\begin{cases}C_1\lambda^{-1}+O(1), & \lambda\ll 1\\ -\pi-a_1\lambda^{-4/3}+O(\lambda^{-8/3}), & \lambda\gg 1,\end{cases}$$

where

$$C=8\pi\left\{\int_1^\infty\mathrm{d}x\left[\frac{1}{Q(x)}-\frac{1}{2\pi(2x)^{1/2}}\right]-\frac{1}{\pi\surd 2}\right\}\approx-5.52;\quad Q(x)=4[2(1+x)]^{1/2}\mathrm{E}\left[\left(\frac{2}{1+x}\right)^{1/2}\right],$$
$$C_1=-\frac{64}{9\pi}(I_e+I_i)\approx-2.72;$$
$$I_e=\int_0^1\frac{\mathrm{d}q}{q^8\,\mathrm{E}(q)}[(2-q^2)\,\mathrm{E}(q)-2(1-q^2)\,\mathrm{K}(q)]^2\approx 0.34;$$
$$I_i=\int_0^1\mathrm{d}q\,q\frac{[(2q^2-1)\,\mathrm{E}(q)-(1-q^2)\,\mathrm{K}(q)]^2}{\mathrm{E}(q)-(1-q^2)\,\mathrm{K}(q)}\approx 0.77;$$
$$a_1=\tfrac{1}{6}\pi\left(\tfrac{3}{2}\right)^{1/3}\Gamma\left(\tfrac{1}{3}\right)\approx 1.6057;$$

E($q$) and K($q$) are complete elliptic integrals.

The function $\Phi_1$ was introduced by Haberman (1972); subsequently, it has been calculated by many authors. The function $\Phi_2$ (in limiting cases $\lambda \gg 1$ and $\lambda \ll 1$) was calculated by Shukhman (1989), and in the limit $\lambda \ll 1$ by GH. Upon substituting (A 7) into (A 6$a$, $b$) and dropping $\Theta$, we get a NEE in the form

$$\frac{\mathrm{d}B}{\mathrm{d}\zeta} = -\frac{\Phi_1(\lambda)}{H(\lambda)}\left\{2\frac{w_c'''}{w_c'^2}I_2\Omega - \eta\zeta\left[\rho\left(2kI_2 - \frac{c}{k}I_1\right) + \frac{cw_c'''}{kw_c'^2}I_3\right]\right\}B, \qquad \text{(A 8)}$$

where $$H(\lambda) = \left(2kI_2 - \frac{c}{k}I_1\right)^2 + \left(\frac{cw_c'''}{kw_c'^2}\right)^2 \Phi_1(\lambda)\,\Phi_2(\lambda).$$

For illustration, it is convenient to cast (A 8) in terms of 'physical' variables: $A = \epsilon B$, $z = \epsilon^{1/2}\zeta$:

$$\frac{\mathrm{d}A}{\mathrm{d}z} = \Phi\left(\frac{\nu}{A^{3/2}}\right)(\gamma_L - \sigma\nu z)A, \quad \Phi(\lambda) = \begin{cases} 1, & \lambda \gg 1 \\ b\lambda, & \lambda \gg 1. \end{cases} \qquad \text{(A 9)}$$

A connection of the function $\Phi$ with $\Phi_1$ and $\Phi_2$, and also an explicit expression for constant $b$ are readily obtained from (A 8); they are not given here because they are too unwieldy.

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